UNIVERSAL CHARACTERISTIC FACTORS AND FURSTENBERG AVERAGES

T. ZIEGLER

Abstract. Let $X = (X^0, \mathcal{B}, \mu, T)$ be an ergodic probability measure preserving system. For a natural number k we consider the averages

$$(*) \qquad \frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} f_j(T^{a_j n}x)$$

where $f_j \in L^\infty(\mu)$, and a_j are integers. A factor of X is characteristic for averaging schemes of length k (or k-characteristic) if for any non zero distinct integers $a_1, \ldots, a_k$, the limiting $L^2(\mu)$ behavior of the averages in (*) is unaltered if we first project the functions f_j onto the factor. A factor of X is a *k-universal characteristic factor (k-u.c.f)* if it is a k-characteristic factor, and a factor of any k-characteristic factor. We show that there exists a unique k-u.c.f, and it has a structure of a $(k-1)$-step nilsystem, more specifically an inverse limit of $(k-1)$-step nilflows. Using this we show that the averages in (*) converge in $L^2(\mu)$. This provides an alternative proof to the one given in Host and Kra [HK02c].

1. Introduction

Averages of the form

$$(1) \qquad \frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} f(T^{jn}x)$$

were first introduced by Furstenberg [Fu77] in his ergodic theoretic proof of Szemerédi's theorem on arithmetic progressions in sets of positive density in $\mathbb{Z}$. Furstenberg proved the following theorem:

1.1. Theorem (Furstenberg). *Let $X = (X^0, \mathcal{B}, \mu, T)$ be a measure preserving system (m.p.s.). Let A be a set of positive measure, and $f = 1_A$. Then*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int\prod_{j=0}^{k} f(T^{jn}x)d\mu > 0.$$

The theorem above ensures that there exists an integer n, such that the points $x, T^n x, \ldots, T^{kn}x$ are in A, and corresponds to the existence of an arithmetic progression of length $k+1$ in sets of positive density in $\mathbb{Z}$.

2000 *Mathematics Subject Classification.* 37Axx.

The L^2 limiting behavior of the averages in equation (1) is related to a natural series of factors of the measure preserving system X. The factor corresponding to arithmetic progression of length 3 (the case where k is 2) - the Kronecker factor - was described in [Fu77]. The factor corresponding to arithmetic progression of length 4, an inverse limit of 2-step nilflows, was studied by Conze - Lesigne [CL84],[CL87],[CL88], Furstenberg - Weiss [FuW96], and Host - Kra [HK01],[HK02a], and hinted to the nature of the complete series. The complete series of factors was discovered by Host-Kra [HK02c], and independently, though somewhat later, by the author. We give an equivalent definition of the series of factors described in [HK02c], and provide a different construction for these factors. Although there are some similarities between the constructions (for example, both start out with the Furstenberg structure theorem [Fu77]), the definition of the factors and the bulk of the construction are significantly different. In particular, we do not use the Gowers uniformity norms [G01], which are fundamental in the approach of Host-Kra to this problem (and in the works of Gowers [G01], and Green-Tao [GT04] on problems of a similar nature). The averages studied in the paper are of a special kind, but the techniques developed can be used in analyzing other multiple averages (e.g. averages along a polynomial sequence).

Let $X = (X^0, \mathcal{B}_X, \mu_X, T_X)$ be a probability measure preserving system; i.e. $(X^0, \mathcal{B}_X, \mu_X)$ is a measure space, and T_X is a measure preserving transformation. When there is no confusion we will omit the subscript X. We write Tf for the function $Tf(x) = f(Tx)$. By ergodic decomposition it will suffice to study the limit of (1) with the additional hypothesis of ergodicity. The nature of the limit will depend on mixing properties of the system. The maximal degree of mixing relevant is *weak mixing*; indeed in this case Furstenberg has shown in [Fu77]:

1.2. **Theorem** (Furstenberg). *If X is weak mixing then*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} f_j(T^{jn}x) \overset{L^2(X)}{\longrightarrow} \prod_{j=1}^{k}\int f_j(x)d\mu.$$

For a general ergodic system X the averages in equation (1) need not converge to a constant function. Indeed, if the system X is not weakly mixing there exists a nontrivial eigenfunction ψ. If $T\psi(x) = \lambda\psi(x)$ then

$$T^n\psi^2(x)T^{2n}\psi^{-1}(x) = \psi(x)$$

for all n, thus

$$\frac{1}{N}\sum_{n=1}^{N} T^n\psi^2(x)T^{2n}\psi^{-1}(x) = \psi(x).$$

By the above equation, the set of limiting functions contains the algebra spanned by eigenfunctions - the *Kronecker algebra*. The Kronecker algebra determines the 'Kronecker factor' Z where Z^0 is a compact Abelian group,

$\mathcal{B}_Z$ the (completed) Borel algebra, μ_Z the Haar measure, and the action of T_Z is given by translation by an element $\alpha \in Z^0$, i.e $T_Z z = z + \alpha$. Let $\pi : X \to Z$ be the factor map. It is not surprising that an Abelian group factor should come up when studying the relations between $x, T^n x, T^{2n} x$, as the projections of these points on the Abelian group factor $\pi(x), \pi(x) + n\alpha, \pi(x) + 2n\alpha$ form an arithmetic progression: $\pi(x) = 2(\pi(x) + n\alpha) - (\pi(x) + 2n\alpha)$. It turns out that this 'constraint' imposed by the Kronecker factor is the only 'constraint' on the triple $x, T^n x, T^{2n} x$, and in a manner to be made precise, the Kronecker factor is 'characteristic' for the limit of the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x) g(T^{2n} x)$.

Let X be a measure preserving system (m.p.s). Let Y be a homomorphic image; i.e, we have a map $\pi : X^0 \to Y^0$ with $\pi^{-1}\mathcal{B}_Y \subset \mathcal{B}_X$, $\pi\mu_X = \mu_Y$ and $T_Y \pi = \pi T_X$. Then Y is a *factor* of X, X is an *extension* of Y, and abusing the notation we write $\pi : X \to Y$ for the factor map. A factor of X is determined by a T_X invariant subalgebra of $L^\infty(\mu)$. The map π induces two natural maps $\pi^* : L^2(\mu_Y) \to L^2(\mu_X)$ given by $\pi^* f = f \circ \pi$, and $\pi_* : L^2(\mu_X) \to L^2(\mu_Y)$ given by $\pi_* f = E(f|\mathcal{B}_Y)$ (the orthogonal projection of f on $\pi^* L^2(\mu_Y)$). We fix an ergodic m.p.s X. The notion of 'characteristic factors' was first introduced in a paper by Furstenberg and Weiss [FuW96].

1.3**. Definition.** Let Y be a factor of X. Let k be a natural number, $(a_1, \dots, a_k)$ be distinct non-zero integers. The system Y is *characteristic for* $(a_1, \dots, a_k)$ if for any $f_1, \dots, f_k \in L^\infty(\mu_X)$,

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} T_X^{a_j n} f_j - \pi^* \frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} T_Y^{a_j n} \pi_* f_j \xrightarrow{L^2(\mu_X)} 0.$$

The system Y is a *k-characteristic* factor of X if it is characteristic for any k-tuple of distinct non-zero integers.

It is in this sense that the Kronecker factor is characteristic for calculating the limit of the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x) g(T^{2n} x)$. We now define a universal characteristic factor:

1.4**. Definition.** Let Y be a factor of X. The system Y is a *k-universal characteristic factor (u.c.f)* of X if it is a k-characteristic factor of X , and a factor of any other k-characteristic factor of X.

For the averages $\frac{1}{N}\sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) f_3(T^{3n} x)$, the Kronecker factor does not suffice. Let φ be a *second order eigenfunction*, i.e., $T\varphi = \psi\varphi$, and $T\psi = \lambda\psi$. Then one can check that

$$T^n \varphi^3(x) T^{2n} \varphi^{-3}(x) T^{3n} \varphi(x) = \varphi(x)$$

for all n; thus

$$\frac{1}{N}\sum_{n=1}^{N} T^n \varphi^3(x) T^{2n} \varphi^{-3}(x) T^{3n} \varphi = \varphi(x). \tag{2}$$

Let Y be a factor of X that is characteristic for $(1,2,3)$. Equation (2) implies that the algebra generated by all second order eigenfunctions is a subset of $L^\infty(\mu_Y)$. It is natural to conjecture that the algebra generated by the second order eigenfunctions determines a factor that is characteristic for $(1,2,3)$. Furstenberg and Weiss presented the following counter example. Let

$$X = \left(\begin{smallmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{smallmatrix}\right) \Big/ \left(\begin{smallmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{smallmatrix}\right) = N/\Gamma.$$

Consider the system X where $X^0 = N/\Gamma$, $\mathcal{B}_X$ the (completed) Borel algebra, μ_X the unique measure invariant under translations by any element of the group N, and T_X is given by $T_X g\Gamma = ag\Gamma$ for some $a \in N$ acting ergodically. This system has no second order eigenfunctions, but there are relations between $g\Gamma, a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$ not coming from Kronecker factor: In N/Γ, $g\Gamma$ is determined by $a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$.

This system can be viewed as a circle extension of the Kronecker factor which is a 2 dimensional torus, and the action of T_X on $\mathbb{T}^2 \times S^1$ is given by $T_X(z,\zeta) = (z+\alpha, \sigma(z)\zeta)$ (the function $\sigma(z)$ is called the *extension cocycle*). The projection of the points $g\Gamma, a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$ on the Kronecker factor will form an arithmetic progression, but as $g\Gamma$ is a function of $a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$ the points $g\Gamma, a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$ will not be independent on the fibers over the Kronecker factor. This fact translates to a restriction on the extension cocycle $\sigma(z)$ known as the Conze-Lesigne equation. (this equation is analyzed in [CL84], [CL87], [CL88], [Le84], [Le87], [Le93], [FuW96], [HK01] [HK02a], [Me90],[R93]). In particular any factor that is characteristic for $(1,2,3)$ will contain functions other than first and second order eigenfunctions.

In general, if N is a k-step nilpotent group ($N_{k+1} = 1$), $\Gamma < N$ then $x \in N/\Gamma$ is determined by $a^n x, a^{2n} x, \dots, a^{(k+1)n} x$.

It is natural to ask whether these are the only constraints, i.e. do *all* the constraints on the points $x, T^n x, \dots, T^{(k+1)n} x$ come from a k-step nilpotent factor?

1.5. **Definition.** A *nilsystem* consists of a space X on which a nilpotent group N acts transitively preserving a measure μ_X, and a transformation T_X which acts by translation by a group element a: $T_X x = ax$. A special case is when N is a k-step nilpotent Lie group, Γ a cocompact lattice, $X = N/\Gamma$ (a nilmanifold), and μ_X the unique measure invariant under translation by elements of N. We call this a k-step *nilflow.* A *k-step pro-nilflow* is an inverse limit of k-step nilflows.

We prove the following theorems:

1.6. **Theorem.** *Let X be an ergodic measure preserving system. Then there exists a unique k-universal characteristic factor of X. If $\pi : X \to Y$ is a factor map, and $W(X), W(Y)$ are k-universal characteristic factors of X, Y respectfully, then π induces a map between $W(X)$ and $W(Y)$.*

If we denote by $Y_k(X)$ the k-u.c.f of X, then one obtains an inverse series of factors $\ldots \to Y_k(X) \to Y_{k-1}(X) \to \ldots \to Y_1(X)$.

1.7. **Theorem.** *Let X be an ergodic measure preserving system, and let $Y_k(X)$ be the k-universal characteristic factor of X. Then $Y_k(X)$ has a structure of a $(k-1)$-step nilsystem, more specifically a $(k-1)$-step pro-nilflow.*

The proof of Theorem 1.7 is by induction: assuming that the k-u.c.f is a $(k-1)$-step nilsystem, one reduces the problem of determining the $(k+1)$-u.c.f to the case where the system X is a circle extension of a $(k-1)$- step nilsystem. If the points $x, T^n x, \ldots T^{(k+1)n}x$ are independent on the fibers over the k-step nilsystem then the k-step nilsystem would suffice, i.e would be k-characteristic. Otherwise one would get a restriction on the extension cocycle. The main difficulty is using this restriction to construct a nilpotent group acting transitively on X.

As a corollary we get a theorem proved recently by Host and Kra [HK02c]

1.8. **Corollary.** *Let X be a m.p.s. Let k be a natural number, $a_1, \ldots a_k \in \mathbb{Z}$, and $f_1 \ldots f_k \in L^\infty(\mu_X)$ then the averages*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{k} f_j(T^{a_j n}x) \tag{3}$$

converge in $L^2(\mu_X)$.

By Theorem 1.7, in order to have $L^2(\mu_X)$ convergence of the averages in (3), it is enough to prove an $L^2(\mu_X)$ convergence theorem for k-step pro-nilflows. Convergence in L^2 for pro-nilflows follows from convergence for nilflows. For nilflows one has a.e. convergence ([P69], [Sh96], [L02]). An explicit description of the limit is given in [Le89] for the case $k = 3$, and in general in [Z02a].

Acknowledgment Most of the ideas in this work appear in the authors PhD thesis. I would like to thank my adviser Prof. Hillel Furstenberg for introducing me to ergodic theory, specifically to questions involving non-conventional ergodic averages, and for many fruitful discussions. I would also like to thank Benji Weiss, Shahar Mozes, Vitaly Bergelson for enlightening conversations and valuable remarks. I owe special thanks to Sasha Leibman for pointing out many inaccuracies in the early version of this paper.

2. Universal characteristic factors

We start by proving Theorem 1.6.

2.1. **Lemma.** *Let X be an ergodic m.p.s. Let Y_1, Y_2 be k-characteristic factors of X. Then there exists a k-characteristic factor of X, which is a factor of both Y_1, Y_2.*

Proof. Denote P, Q the orthogonal projections onto $L^2(\mu_{Y_1}), L^2(\mu_{Y_2})$ (seen as subspaces of $L^2(\mu_X)$) respectfully, and by $\pi_i : X^0 \to Y_i^0$ for $i = 1, 2$ the factor maps. Then $P^2 = P^* = P$ (same for Q). We show that $(PQP)^n$ strongly converges to a self adjoint operator projection W: P is a projection thus $P \leq I$.

$$\left\langle (PQP)^2 x, x \right\rangle = \langle PQPx, QPx \rangle \leq \langle QPx, QPx \rangle = \langle PQPx, x \rangle ,$$

Inductively, the sequence $(PQP)^n$ is a decreasing sequence of operators, thus $\langle (PQP)^n x, x \rangle$ converges for all x. The sequence $(PQP)^n x$ is a Cauchy sequence as

$$\begin{aligned} \|(PQP)^n x - (PQP)^m x\|^2 = &\left\langle (PQP)^{2n} x, x \right\rangle + \left\langle (PQP)^{2m} x, x \right\rangle \\ &- 2 \left\langle (PQP)^{(n+m)} x, x \right\rangle \to 0. \end{aligned}$$

Let $W = \lim_{n\to\infty} (PQP)^n$, then $W^2 = W = W^*$. If $Wx = x$ then $Px = PWx = Wx = x$, and

$$PQx = PQPx = PQPWx = Wx = x \Rightarrow \|Qx\| = \|x\| \Rightarrow Qx = x.$$

It follows that $W(L^2(X^0, \mathcal{B}_X, \mu_X)) = L^2(X^0, \mathcal{D}, \mu_X)$ for $\mathcal{D} = \pi_1^{-1}(\mathcal{B}_{Y_1}) \cap \pi_2^{-1}(\mathcal{B}_{Y_2})$. We show that $W(L^2(\mu_X))$ is a k-characteristic factor of X. For all m:

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_1 n} f_1 \ldots T^{a_k n} f_k \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_1 n}((PQP)^m f_1) \ldots T^{a_k n}((PQP)^m f_k) \end{aligned}$$

□

2.2. **Corollary.** *Let X be a m.p.s. There exists a unique k-universal characteristic factor of X.*

Proof. By Zorn's lemma. □

The advantage of looking at all k-tuples (rather than focusing on a specific one) is that k-u.c.f are natural in the sense that any morphism of measure preserving systems induces a morphism between their k-universal characteristic factors - as will be shown in corollary 2.4. (This may also be true for characteristic factors of a specific scheme).

2.3. **Lemma.** *Let V be the algebra generated by partial limits of the sequences $\{\frac{1}{N}\sum_{n=1}^{N} T^{a_1 n} f_1 \ldots T^{a_k n} f_k\}$, where $f_i \in L^\infty(\mu_X)$, and $a_0, \ldots, a_k \in \mathbb{Z}$ are distinct non zero integers. Then V determines the k-universal characteristic factor of X.*

Proof. Let $W(X)$ be the k-universal characteristic factor. Obviously $V \subset L^\infty(\mu_{W(X)})$. We show that the factor determined by V is a k-characteristic factor of X. Let $g \perp V$, then for any f_1

$$\begin{aligned}\left\langle g, \frac{1}{N}\sum_{n=1}^{N} T^{a_1 n} f_1 \dots T^{a_k n} f_k \right\rangle =& \frac{1}{N}\sum_{n=1}^{N} \int g T^{a_1 n} f_1 T^{a_2 n} f_2 \dots T^{a_k n} f_k d\mu \\ =& \frac{1}{N}\sum_{n=1}^{N} \int f_1 T^{-a_1 n} g \dots T^{(a_k - a_1) n} f_k d\mu \\ & \stackrel{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

2.4. **Corollary.** *If $\pi : X \to Y$ is a factor map, and $W(X), W(Y)$ are k-u.c.f. for X, Y respectfully, then π induces a map between $W(X)$ and $W(Y)$.*

2.5. **Universal characteristic factors for $k = 1, 2$.** Let X be an ergodic m.p.s. If the system X is totally ergodic then by von Neumann's theorem the trivial system (a point) would be the 1-u.c.f of X. Otherwise one needs to take into account the algebra generated by functions that are invariant under T_X^m for some m. The factor may then be represented as a finite cyclic group with addition of one, or a pro-cyclic group which is an inverse limit of cyclic groups. The 2-universal characteristic factor of X coincides with the first block in Furstenberg's structure theorem (see [Fu77]) and is referred to as *the Kronecker factor*. The system Z is a *Kronecker system* (or an *almost periodic system*) if Z^0 is a compact abelian group (a 1-step nilpotent group), $\mathcal{B}_Z$ is the (completed) Borel algebra, μ_Z is the Haar measure, and the action of T_Z is given by $T_Z z = z + \alpha$ for some $\alpha \in Z^0$. The Kronecker factor is the maximal almost periodic factor. Equivalently, Z is the Kronecker factor of X if the eigenfunctions of T_X span $L^2(\mu_Z)$ (thought of as a subspace of $L^2(\mu_X)$).

2.6. **Remark.** If the system X is weak mixing, i.e. has no non trivial eigenfunctions, then the Kronecker factor is trivial (and $Y_k(X)$ is trivial for all k).

2.7. **Isometric extensions** The notion of characteristic factors was motivated by Furstenberg's structure theorem [Fu77]. Furstenberg's idea was to relativize the notion of weak mixing to a *weak mixing extension* and to define the complementary notion of a *compact extension* (or *isometric extension*). Let X be an ergodic m.p.s., and let Y be a factor. Consider the ring $L^\infty(\mu_Y)$ as a subring of functions on X. A subspace $V \subset L^2(\mu_X)$ is a *finite rank module* over $L^\infty(\mu_Y)$ if there exist finitely many functions $\varphi_1, \dots, \varphi_k$, such that any function $f \in V$ can be expressed as $f = \sum_{i=1}^k a_i(y)\varphi_i(x)$. We say that X is an *isometric extension* of Y if $L^2(\mu_X)$ is spanned by finite rank T_X invariant modules over $L^\infty(\mu_Y)$. It can be shown that in this case X is isomorphic to a *skew product* X' where $X'^0 = Y^0 \times M$, where $M = G/H$ is

a homogeneous compact metric space, $\mu_{X'} = \mu_Y \times m_M$, where m_M is the unique probability measure invariant under the transitive group of isometries G, and the action of $T_{X'}$ is given by $T_{X'}(y,m) = (T_Y y, \rho(y)m)$, where $\rho : Y^0 \to G$. We denote $T_{X'}$ by $T_{Y,\rho}$, or if there is no confusion, just T_ρ. For example, a Kronecker system is an isometric extension of a point. Define $\rho^{(n)} : Y^0 \to G$ by $T^n_\rho(y,m) = (T^n y, \rho^{(n)}(y)m)$; then $\rho^{(n)}$ satisfies a 1-cocycle equation for the action of $\mathbb{Z}$ on functions from Y to G

$$\rho^{(n+m)}(y) = \rho^{(n)}(T^m y)\rho^{(m)}(y).$$

Since $\rho^{(n)}(y)$ is determined by $\rho^{(1)}(y)$ we shall focus on $\rho(y) = \rho^{(1)}(y)$ and refer to it as the *extension cocycle* (or just *cocycle*). Abusing the notation we denote the system X' by $Y \times_\rho G/H$. For more details see [Fu77], or [Zi76]

2.8. Let X_1, X_2 be m.p.s. and let Y be a common factor with $\pi_i : X_i^0 \to Y^0$ for $i = 1,2$ the factor maps. Let $\mu_{X_{i,y}}$ represent the disintegration of μ_{X_i} with respect to Y. Denote $\mu_{X_1} \times_Y \mu_{X_2}$ the measure defined by

$$\mu_{X_1} \times_Y \mu_{X_2}(A) = \int \mu_{X_{1,y}} \times_Y \mu_{X_{2,y}}(A) d\mu_Y$$

for $A \in \mathcal{B}_{X_1} \times \mathcal{B}_{X_2}$. The system

$$(X_1^0 \times X_2^0, \mathcal{B}_{X_1} \times \mathcal{B}_{X_2}, \mu_{X_1} \times_Y \mu_{X_2}, T_{X_1} \times T_{X_2})$$

is called the relative product of X_1 and X_2 with respect to Y and is denoted $X_1 \times_Y X_2$.

2.9. Let X be an ergodic m.p.s., Y a factor and $\pi : X \to Y$ the factor map. Consider the subspace of $L^2(\mu_X)$ spanned by all finite rank T_X-invariant modules over $\pi^* L^\infty(\mu_Y)$. This subspace will be defined by some factor $\hat{Y}$ between X and Y. The system $\hat{Y}$ is called the *maximal isometric extension of* Y *in* X. For some $l \in BN$, let $X' = (X^0, \mathcal{B}_X, \mu_X, T_X^l)$, and let $Y' = (Y^0, \mathcal{B}_Y, \mu_Y, T_Y^l)$. Then the maximal isometric extension of Y' in X' is $\hat{Y}' = (\hat{Y}^0, \mathcal{B}_{\hat{Y}}, \mu_{\hat{Y}}, T_{\hat{Y}}^l)$.

2.10. Let X_i, $i = 1, \ldots, k$, be measure preserving systems, and let Y_i be corresponding factors, and $\pi_i : X_i^0 \to Y_i^0$ the factor maps. A measure ν on ΠY_i^0 defines a *joining* of the measures on Y_i if it is invariant under $T_{Y_1} \times \ldots \times T_{Y_K}$ and maps onto ν_{Y_i} under the natural map $\Pi Y_i \to Y_j$. Let ν be a joining of the measures on Y_i, and let $\mu_{X_{i,y_i}}$ represent the disintegration of μ_{X_i} with respect to Y_i.

Let μ be a measure on ΠX_i^0 defined by

$$\mu = \int \mu_{X_{1,y_1}} \times \ldots \times \mu_{X_{k,y_k}} d\nu(y_1, \ldots, y_k).$$

Then μ is called the *conditional product measure* with respect to ν.

The following is shown in [Fu77] Theorem 9.5:

2.11. **Theorem** (Furstenberg). *Let X_i, Y_i, ν, μ be as in 2.10. Assume each X_i has finitely many ergodic components. Let $\hat{Y}_i$ be the maximal isometric extension of Y_i in X_i, $\hat{\pi}_i : X_i \to \hat{Y}_i$ the projection. Then if $F \in L^2(\mu)$ is invariant under $T_{X_1} \times \ldots \times T_{X_k}$ then there exists a function $\Phi \in L^2(\mu)$, so that*

$$F(x_1, \ldots, x_k) = \Phi(\hat{\pi}_1(x_1), \ldots, \hat{\pi}_k(x_k)).$$

2.12. **Group extensions** A special case of an isometric extension $X \to Y$ is when the homogeneous space M from 2.7 is equal to G, i.e $X = Y \times_\rho G$ where G is a compact group. In this case we say that X is a *group extension* of Y.

2.13. **Lemma.** *Suppose X is an ergodic isometric extension of Y so that we can express $X = Y \times_\rho G/H$. Using the function ρ, we can define a group extension $Y \times_\rho G$. Then G and H can be chosen so that the extension $Y \times_\rho G$ is an ergodic group extension.*

Proof. [FuW96] lemma 7.2. □

2.14. **Lemma.** *Let $X = Y \times_\rho G$ be an ergodic group extension of Y, and let W be an intermediate factor between X and Y, then X is a group extension of W.*

Proof. [FuW96] lemma 7.3 . □

2.15. Let Y be an ergodic m.p.s., G a compact metrizable group. Let $Y \times_\rho G$ be a group extension. We can parameterize $Y^0 \times G$ replacing (y, g) with $F(y, g) = (y, f(y)g)$ for some measurable function $f : Y^0 \to G$. Let $\rho'(y) := f(Ty)\rho(y)f(y)^{-1}$, then the systems $Y \times_\rho G$, $Y \times_{\rho'} G$ are isomorphic, and ρ, ρ' are called *equivalent cocycles* or *cohomologous cocycles*. If ρ is equivalent to the identity cocycle then ρ is a Y*-coboundary* (or just *coboundary* when there is no confusion). If ρ is equivalent to a constant cocycle then ρ is a Y*-quasi-coboundary* (or *quasi-coboundary*).

2.16. If ρ takes values in a closed subgroup H of G, the extension $Y \times_\rho G$ will not be ergodic (any function on $H\backslash G$ will be invariant). By the foregoing discussion if ρ is equivalent to a cocycle taking values in a closed subgroup H, then the extension $Y \times_\rho G$ will not be ergodic.

2.17. **Theorem** (Mackey). *Let $\rho : Y^0 \to G$ be a measurable cocycle. There exists a closed subgroup $M < G$, unique up to conjugacy, so that:*

(1) *ρ is equivalent to a cocycle ρ' taking values in M.*
  i.e. $\rho'(y) = f(Ty)\rho(y)f(y)^{-1} \in M$.
(2) *Any ergodic $T_{\rho'}$-invariant measure on $Y^0 \times G$, extending μ_Y, has the form $\mu_Y \times m_{M\gamma}$ for some coset $M\gamma$, and the ergodic T_ρ invariant measures are obtained by applying F^{-1} (defined in 2.15 to the ergodic $T_{\rho'}$-invariant measures. The group M is called the* Mackey group *of the extension $Y \times_\rho G$.*

2.18. **Lemma.** *For $i = 1, 2$, let Y_i be ergodic m.p.s, let $X_i = Y_i \times_{\rho_i} G$ be group extensions, and let M_i be the associated Mackey groups. Let π_i be the projection $\pi_i : X_i \to Y_i$. Let $S : X_1 \to X_2$ be an isomorphism such that $S : Y_1 \to Y_2$ ans $S\pi_1 = \pi_2 S$. Then M_1 and M_2 are conjugate.*

Proof. The transformation S maps the ergodic components of the group extension $Y_1 \times_{\rho_1} G$ onto those of $Y_2 \times_{\rho_2} G$. Each ergodic component is determined by a right coset $M_i\gamma_i$ for $\gamma_i \in G$, thus S induces a map from $\varphi : M_1\backslash G \to M_2\backslash G$, that commutes with the action of G from the right. Thus φ is a G-isomorphism, and therefore M_1 and M_2 are conjugate. □

3. ABELIAN EXTENSIONS

3.1. **Notation.** We use additive notation for abelian groups with the exception of the group $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ which will play a special role in the future. In particular, if ρ, ρ' are equivalent cocycles (defined in the foregoing section) taking values in an abelian group G, then there exists a function $f : Y^0 \to G$ such that

$$\rho(y) = f(Ty) + \rho'(y) - f(y).$$

3.2. Let G be a compact abelian group, then $Y \times_\rho G$ is an *abelian extension.* In this case the Mackey group defined in the foregoing section M is unique. Let

$$M^\perp = \{\chi \in \hat{G} : \chi(g) = 1 \text{ for all } g \in M\}$$

be the annihilator of M. If ρ is equivalent to a cocycle taking values in M then $\chi \circ \rho$ is a coboundary for all $\chi \in \hat{M}$.

$$M^\perp = \{\chi \in \hat{G} : \chi \circ \rho \text{ is a coboundary}\}.$$

3.3. **Proposition.** *Let $Y \times_\rho G$ be an abelian extension, and let M be the Mackey group of this extension. Let $f \in L^2(\mu_Y \times m_G)$ be s.t. for all $\chi \in M^\perp$,*

$$\int f(y, g)\chi(g)dm_G(g) = 0$$

for a.e $y \in Y$. Then f is orthogonal to the space of T_ρ invariant functions.

Proof. [FuW96] lemma 9.2. □

3.4. **Notation.** Denote $U_d = d$ dimensional unitary matrices, $C(U_d)$ the center of U_d (scalar matrices), and $P : U_d \to \mathbb{P}U_d = U_d/C(U_d)$ the natural projection. For $U, V \in U_d$ denote by $[U, V]$ the commutator of U, V; i.e., $[U, V] = UVU^{-1}V^{-1}$.

We need the following lemma:

3.5. **Lemma.** *Let H be a compact abelian connected group, and $A : H \to U_d$ a measurable function. If $P \circ A$ is a homomorphism, then $A(H)$ is a commuting set of matrices.*

Proof. Let $g, h \in H$. Suppose $[A(h), A(g)] = \delta I$. If v is an eigenvector of $A(h)$ with eigenvalue γ, then

$$A(h)A(g)v = \delta A(g)A(h)v = \gamma\delta A(g)v$$

thus $A(g)v$ is an eigenvector of $A(h)$ with eigenvalue $\gamma\delta$. This implies that $A(g)^k v$ is an eigenvector of $A(h)$ with eigenvalue $\gamma\delta^k$, thus δ is a root of unity of order $\leq d$. Denote $C_{d!}$ the group of order $d!$ roots of unity. Then the commutator set

$$\{[A(h), A(g)]\}_{h,g\in H} \subset C_{d!}I,$$

Fix g. The function $h \to [A(h), A(g)]$ is a measurable homomorphism to $C_{d!}$

$$\begin{aligned}[A(h_1 + h_2), A(g)] &= [cA(h_1)A(h_2), A(g)] \\ &= [A(h_1), A(g)][A(h_2), A(g)],\end{aligned}$$

therefore continuous, and as H is connected it is trivial. □

3.6**. Theorem.** *Let Y be an ergodic m.p.s, and let $W = Y\times_\rho H$ be an ergodic extension by a connected abelian group. Let $F : Y^0 \times H \times H \to S^1$ be a measurable function. Let $\sigma_1(y, h_1)$,$\sigma_2(y, h_2) : Y^0 \times H \to S^1$ be measurable functions. Suppose*

$$\sigma_1(y, h_1)\sigma_2(y, h_2) = \frac{F(Ty, h_1 + \rho(y), h_2 + \rho(y))}{F(y, h_1, h_2)}.$$

Then for $i = 1, 2$ there exist measurable functions $g_i, G_i : Y^0 \to S^1$ such that

$$\sigma_i(y, h) = g_i(y)\frac{G_i(T_W(y, h))}{G_i(y, h)}$$

Proof. We construct the following systems: for $i = 1, 2$ let $X_i = W \times_{\sigma_i} S^1$, and $X = X_1 \times_Y X_2$. Then μ_X is defined as the conditional product measure relative to the diagonal measure on $Y^0 \times Y^0$. The function

$$\tilde{F}(y, h_1, h_2, \zeta_1, \zeta_2) = F(y, h_1, h_2)\zeta_1^{-1}\zeta_2^{-1} \tag{4}$$

is invariant under T_X, and therefore by Theorem 2.11 it is measurable with respect to $\hat{Y}_1 \times \hat{Y}_2$, where $\hat{Y}_i$ is the maximal isometric extension of Y in X_i for $i = 1, 2$. Isometric extensions are spanned by finite rank modules (see 2.7). Thus

$$\tilde{F}(y, h_1, h_2, \zeta_1, \zeta_2) = \sum \left\langle \vec{\psi_j^1}(y, h_1, \zeta_1), \vec{\psi_j^2}(y, h_2, \zeta_2)\right\rangle$$

where

$$\begin{aligned}T_{X_1}\vec{\psi_j^1}(y, h_1, \zeta_1) &= U_j^1(y)\vec{\psi_j^1}(y, h_1, \zeta_1) \\ T_{X_2}\vec{\psi_j^2}(y, h_2, \zeta_i) &= U_j^2(y)\vec{\psi_j^2}(y, h_2, \zeta_2),\end{aligned}$$

and $U_j^1(y), U_j^2(y)$ are $d_j \times d_j$ unitary matrices. Substituting the Fourier expansions:

$$\vec{\psi}_j^1(y, h_1, \zeta_1) = \sum \vec{\psi}_{j,k}^1(y, h_1)\zeta_1^k$$
$$\vec{\psi}_j^2(y, h_2, \zeta_2) = \sum \vec{\psi}_{j,k}^2(y, h_2)\zeta_2^k$$

in equation(4) we get that for $k = -1$ there exists j such that $\vec{\psi}_{j,-1}^1 \neq 0$. Apply T_{X_1} to get

$$\sigma_1^{-1}(y, h_1)\, \vec{\psi}_{j,-1}^1(T_W(y, h_1)) = U_j^1(y)\, \vec{\psi}_{j,-1}^1(y, h_1)$$

For simplicity we drop the indices:

$$\sigma^{-1}(y, h)\, \vec{\psi}(T_W(y, h)) = U(y)\, \vec{\psi}(y, h) \tag{5}$$

For each y consider the distribution of $\vec{\psi}(y, h)$ in the fiber over y, and look at the vector space spanned by the support of this distribution. Call this V_y, so that $V_y \subset \mathbb{C}^d$, and $V_{Ty} = U(y)V_y$. Since $U(y)$ is unitary, $dimV_{Ty} = dimV_y$, thus by ergodicity $dimV_y = \hat{d}$ for a.s. y. For each y choose a basis for $\mathbb{C}^d$ s.t. V_y is spanned by the first $\hat{d}$ elements. As the transformation matrix is a function of y, we may assume $d = \hat{d}$.

Denote by $\vec{\tilde{\psi}}$ the projection of $\vec{\psi}$ on $\mathbb{P}V$, and by $\tilde{U}$ the projection of U on $\mathbb{P}U_d$. Thus:

$$\vec{\tilde{\psi}}(T_W(y, h)) = \tilde{U}(y)\, \vec{\tilde{\psi}}(y, h).$$

Consider the group extension $W \times_{\tilde{U}} \mathbb{P}U_d$. Then

$$\vec{\tilde{\psi}}(T_W^n(y, h)) = \tilde{U}^{(n)}(y)\, \vec{\tilde{\psi}}(y, h)$$

For fixed y, $\{\vec{\tilde{\psi}}(y, h)\}_{h \in H}$ spans the space, so whenever $T_W^n(y, h) = (T^n y, h + \rho^{(n)}(y))$ is close to (y, h) (by ergodicity this happens for a generic y, and is independent of h), $\tilde{U}^{(n)}(y)$ is close to the identity. This implies that the foregoing group extension is not ergodic, and furthermore - the Mackey group is trivial. Thus for some projective unitary matrix function $\tilde{M}$:

$$\tilde{M}(T_W(y, h)) = \tilde{U}(y)\tilde{M}(y, h) \tag{6}$$

Also for any h'

$$\tilde{M}(T_W(y, h + h')) = \tilde{U}(y)\tilde{M}(y, h + h').$$

Thus

$$\tilde{M}^{-1}(T_W(y, h + h'))\tilde{M}(T_W(y, h)) = \tilde{M}^{-1}(y, h + h')\tilde{M}(y, h)$$

By ergodicity

$$\tilde{M}^{-1}(y, h + h')\tilde{M}(y, h) = \tilde{A}(h'),$$

for all h', a.e.(y,h). By Fubini's theorem there exists h_0 such that

$$\tilde{M}(y,h) = \tilde{M}(y,h_0)\tilde{A}^{-1}(h-h_0), \tag{7}$$

for a.e.(y,h). The function $\tilde{A}(h')$ is a homomorphism of H:

$$\begin{aligned}\tilde{A}(h'+h'') &= \tilde{M}^{-1}(y,h+h'+h'')\tilde{M}(y,h)\\ &= \tilde{M}^{-1}(y,h+h'+h'')\tilde{M}(y,h+h')\tilde{M}^{-1}(y,h+h')\tilde{M}(y,h)\\ &= \tilde{A}(h'')\tilde{A}(h')\end{aligned}$$

Recall $P : U_d \to \mathbb{P}U_d$ is the natural projection. We can find a measurable function $A : H \to U_d$ so that $P \circ A = \tilde{A}$.

$$A(H) \subset P^{-1}\tilde{A}(H).$$

Then by lemma 3.5, $A(H)$ is a commuting set. Substituting equation (7) in equation (6) we get

$$\begin{aligned}\tilde{M}(Ty,h_0)\tilde{A}^{-1}(h+\rho(y)-h_0) =& \tilde{M}(Ty,h+\rho(y))\\ =& \tilde{U}(y)\tilde{M}(y,h)\\ =& \tilde{U}(y)\tilde{M}(y,h_0)\tilde{A}^{-1}(h-h_0)\end{aligned}$$

Thus

$$\tilde{U}(y) = \tilde{M}(Ty,h_0)\tilde{A}^{-1}(\rho(y))\tilde{M}^{-1}(y,h_0)$$

or

$$U(y) = M(Ty,h_0)A(-\rho(y))M^{-1}(y,h_0)d(y) \tag{8}$$

where $d(y)$ is a scalar matrix. As $A(H)$ is a commuting set, it is simultaneously diagonalizable:

$$A(h) = N^{-1}D(h)N \tag{9}$$

therefore

$$U(y) = M(Ty,h_0)N^{-1}D(-\rho(y))NM^{-1}(y,h_0)d(y)$$

Denote $M'(y) = M(y,h_0)$. Substitute $U(y)$ in equation (5):

$$\sigma^{-1}(y,h)NM'^{-1}(Ty)\overrightarrow{\psi}(Ty,h+\rho(y)) = D(-\rho(y))d(y)NM'^{-1}(y)\overrightarrow{\psi}(y,h)$$

Now each coordinate gives us the desired result. □

3.7. **Remark.** If H in Theorem 3.6 is not necessarily connected, but the cocycle ρ is cohomologous to a constant: $\rho(y) = c\frac{f(Ty)}{f(y)}$, then we do not need to use lemma 3.5, and the result holds as for some scalar matrix $d(y) : Y^0 \to S^1$

$$A(\rho(y)) = A(cf(Ty)f^{-1}(y)) = A(f(Ty))A(c)A^{-1}(f(y))d(y)$$

Now diagonalize $A(c)$: $A(c) = UDU^{-1}$ and substitute in equation (8).

3.8**. Theorem.** *Let* $Y = (Y^0, \mathcal{B}_Y, \mu_Y, T_Y)$ *be an ergodic m.p.s. For* $i = 1, \ldots, k$ *let* $Y_i = (Y^0, \mathcal{B}_Y, \mu_Y, T_Y^i)$. *Let* $W = Y \times_\rho H$ *be an ergodic group extension, where* H *is a connected abelian group, and let* $W_i = Y_i \times_{\rho^{(i)}} H$ *(notice that* $T_{W_i} = T_W^i$*). Let* ν *be a joining of the measures on* Y_i *and, let* μ *be a measure on* ΠW_i^0 *that is the conditional product measure with respect to* ν. *Let* $F : \Pi W_i^0 \to S^1$ *be a* μ *measurable function. For* $i = 1, \ldots, k$, *let* $\sigma_i : W^0 \to S^1$ *be measurable functions, and denote* $\pi : W^0 \to Y^0$ *the projection. Suppose* μ *a.e.*

$$\prod_{i=1}^{k} \sigma_i^{(i)}(w_i) = \frac{F(T_{W_1}w_1, \ldots, T_{W_k}w_k)}{F(w_1, \ldots, w_k)}.$$

Then there exist measurable functions $g_i, G_i : Y^0 \to S^1$ *such that*

$$\sigma_i^{(i)}(w) = g_i(\pi(w))\frac{G_i(T_{W_i}(w))}{G_i(w)} \tag{10}$$

Proof. The proof is similar to the proof of Theorem 3.6. For $i = 1, \ldots, k$ let $X_i = W_i \times_{\sigma_i} S^1$. Let X be the system with $X^0 = \prod X_i^0$, μ_X the conditional product measure with respect to ν, and $T_X = T_{X_1} \times \ldots \times T_{X_k}$. The function

$$\tilde{F}(w_1, \zeta_1, \ldots, w_k, \zeta_k) = F(w_1, \ldots, w_k)\zeta_1^{-1} \ldots \zeta_k^{-1} \tag{11}$$

is invariant under T_X. Proceeding as in Theorem 3.6. we find that equation (10) holds on the ergodic components of T_W^i. As T_W is ergodic, T_W^i has finitely many ergodic components. Let Y_l be an ergodic component of T_Y^i. The ergodic components of T_W^i which project onto Y_l are determined by the Mackey group M_l which is a closed subgroup of H. As T_W^i has finitely many ergodic components M_l is of finite index in H, but H is connected, therefore has no closed subgroups of finite index. Therefore the ergodic components of T_W^i are of the form $Y_l \times H$. □

3.9**. Corollary.** *Let* Y *be an ergodic m.p.s,* $X = Y \times_\rho H$ *an ergodic abelian extension where either* H *is connected or the cocycle* ρ *is cohomologous to a constant. Suppose there exists a measurable family of measurable functions* $\{f_u\}_{u \in H}$, $f_u : Y^0 \times H \to S^1$ *such that*

$$\frac{\sigma(y, h+u)}{\sigma(y, h)} = \frac{f_u(T_X(y, h))}{f_u(y, h)},$$

then there exist measurable functions $g : Y^0 \to S^1$ *and* $F : Y^0 \times H \to S^1$ *such that*

$$\sigma(y, h) = g(y)\frac{G(T_X(y, h))}{G(y, h)}.$$

Proof. Make the coordinate change: $h_1 = h$; $h_2 = h + u$. Then

$$f_u(y, h) = f(y, u, h) = f'(y, h+u, h) = f'(y, h_1, h_2)$$

and

$$f_u(Ty, h + \rho(y)) = f'(Ty, h_1 + \rho(y), h_2 + \rho(y)).$$

Now apply Theorem 3.6. □

3.10. **Lemma.** *Let $Y = Z \times_\rho H$ be an ergodic abelian extension of Z, and $F : Z^0 \times H \to S^1$, $g : Z^0 \to S^1$ measurable functions such that*

$$g(z) = \frac{T_Y F(z,h)}{F(z,h)}.$$

Then there exists $\chi \in \hat{H}$, and $k : Z^0 \to S^1$ such that

$$F(z,h) = k(z)\chi(h).$$

Proof. Take the Fourier expansion of F:

$$F(z,h) = \sum k_i(z)\chi_i(h).$$

Then for all i

$$k_i(T_Z z)\chi_i(h)\chi_i(\rho(z)) = g(z)k_i(z)\chi_i(h).$$

Ergodicity of T_Z implies $|k_i(z)|$ is constant a.e. The fact that $|F| = 1$ implies that there exist an i for which $|k_i(z)| \neq 0$. If there are two such indices i, j, then

$$\frac{\chi_i}{\chi_j}(\rho(z))$$

is a coboundary. As T_Y is ergodic $\chi_i/\chi_j = 1$ (otherwise the Mackey group of the extension $Z \times_\rho H$ is not H). □

3.11. **Notation.** Let $(X_1^0, \mathcal{B}_1)$, $(X_2^0, \mathcal{B}_2)$ be measure spaces. Denote

$$B(X_1^0, X_2^0) = \{f : X_1^0 \to X_2^0, f \text{ measurable}\}.$$

3.12. **Lemma.** *Let $Y = Z \times_\rho H$ be an ergodic abelian extension of Z, (X, μ) a measure space, and let $x \to f_x(y)$ be a Borel measurable function from X to $B(Y^0, S^1)$. Suppose for all $x \in X$ there are functions $g_x(z), F_x(y) \in B(Y^0, S^1)$ such that*

$$f_x(y) = g_x(z)\frac{T_Y F_x(y)}{F_x(y)}. \tag{12}$$

Then there is a μ measurable choice of $g_x(z), F_x(y)$.

Proof. Endowed with the L^2 topology, $B(Y^0, S^1)$ is a polish group. Let $B(Z^0, S^1)$ be the closed subgroup of $B(Y^0, S^1)$ of functions that depend only on the z coordinate, and let $f \to \bar{f}$ be the natural projection onto $\bar{B} = B(Y^0, S^1)/B(Z^0, S^1)$, with the induced topology. By a theorem of Dixmier ([BK96] Theorem 1.2.4) there is a measurable section $\bar{B} \to B$. Equation (12) implies

$$\bar{f}_x(y) = \frac{T_Y \bar{F}_x(y)}{\bar{F}_x(y)}.$$

Define $\varphi : \bar{B} \to \bar{B}$

$$\varphi(\bar{f}) = \frac{T_Y \bar{f}}{\bar{f}}.$$

If $\varphi(\bar{f}) = \varphi(\bar{g})$, then for some function $h(z)$

$$\frac{T_Y \frac{f}{g}(y)}{\frac{f}{g}(y)} = h(z).$$

By 3.10 this implies that up to multiplication by a function of z, $\frac{f}{g}$ belongs to a countable set, thus φ is countable to one. By Lusin [Lu30] $\varphi(\bar{B})$ is a measurable set and there is a measurable function $\psi : \varphi(\bar{B}) \to \bar{B}$ s.t.

$$\varphi \circ \psi = Id|_{\varphi(\bar{B})}$$

Now if

$$\psi(\bar{f}_x) = \bar{F}_x,$$

then

$$\bar{f}_x = \varphi \circ \psi(\bar{f}_x) = \frac{T_Y \bar{F}_x}{\bar{F}_x}.$$

The composition

$$x \to f_x \to \bar{f}_x \to \bar{F}_x \to F_x$$

gives a measurable choice of F_x, and g_x is measurable as a quotient of measurable functions. □

3.13. **Remark.** If $g_x(z) \in B(Y^0, *)$ (g_x is constant) then the same proof works to give a measurable choice of g_x, F_x.

3.14. **Notation.** We write $f \sim g$ if f/g =const.

3.15. **Lemma.** *Let $X = Y \times_\rho H$ be an ergodic abelian extension of Y. Let $\sigma : Y^0 \times H \to S^1$ be such that for all $u \in H$ there exists a measurable function $f_u : Y^0 \times H \to S^1$ and a constant λ_u such that*

$$\frac{\sigma(y, h+u)}{\sigma(y,h)} = \lambda_u \frac{f_u(T_X(y,h))}{f_u(y,h)}, \tag{13}$$

Then there exists a measurable family of measurable functions $\{f_u\}_{u\in H}$, a measurable family of constants $\{\lambda_u\}_{u\in H}$ satisfying the above equation, and a neighborhood of zero U in H such that

$$f_{u_1+u_2}(y,h) \sim f_{u_2}(y, h+u_1) f_{u_1}(y,h)$$
$$\lambda_{u_1+u_2} = \lambda_{u_1}\lambda_{u_2}$$

whenever $u_1, u_2, u_1+u_2 \in U$.

Proof. By remark 3.13 we may assume that the families $\{f_u\}_{u\in H}$, and $\{\lambda_u\}_{u\in H}$ depend measurably on u. Using equation (13) we get

$$\begin{aligned}\frac{\sigma(y, h+u_1+u_2)}{\sigma(y,h)} &= \lambda_{u_1+u_2} \frac{T_X f_{u_1+u_2}(y,h)}{f_{u_1+u_2}(y,h)} \\ &= \lambda_{u_1}\lambda_{u_2} \frac{T_X f_{u_1}(y, h+u_2)}{f_{u_1}(y,h+u_2)} \frac{T_X f_{u_2}(y,h)}{f_{u_2}(y,h)}\end{aligned}$$

this implies that

$$\frac{f_{u_1+u_2}(y,h)}{f_{u_1}(y,h+u_2)f_{u_2}(y,h)}$$

is an eigenfunction of T_X and that

$$\frac{\lambda_{u_1}\lambda_{u_2}}{\lambda_{u_1+u_2}}$$

is an eigenvalue. Let Z be the Kronecker factor of X, $\pi : X^0 \to Z^0$ the projection map, let N parametrize $\hat{Z}$, and let $\psi_{N(u_1,u_2)}(z)$ be a character of Z s.t.:

$$\frac{f_{u_1+u_2}(y,h)}{f_{u_2}(y,h+u_1)f_{u_1}(y,h)} \sim \psi_{N(u_1,u_2)} \circ \pi(y,h) \tag{14}$$

and

$$\frac{\lambda_{u_1}\lambda_{u_2}}{\lambda_{u_1+u_2}} = \psi_{N(u_1,u_2)}(\alpha). \tag{15}$$

Any two characters taking the same value on α are the same, therefore $\psi_{N(u_1,u_2)}$ is symmetric, i.e

$$\psi_{N(u_1,u_2)} = \psi_{N(u_2,u_1)}$$

We now show that $\psi_{N(u_1,u_2)}$ satisfies a 2-cocycle equation:

$$\psi_{N(u_1+u_2,u_3)} \circ \pi(y,h) \sim \frac{f_{u_1+u_2+u_3}(y,h)}{f_{u_3}(y,h+u_1+u_2)f_{u_1+u_2}(y,h)}$$
$$\psi_{N(u_1,u_2+u_3)} \circ \pi(y,h) \sim \frac{f_{u_1+u_2+u_3}(y,h)}{f_{u_2+u_3}(y,h+u_1)f_{u_1}(y,h)}$$

Thus

$$\begin{aligned}(\psi_{N(u_1+u_2,u_3)} \circ \pi(y,h))&f_{u_3}(y,h+u_1+u_2)f_{u_1+u_2}(y,h)\\ &\sim(\psi_{N(u_1,u_2+u_3)} \circ \pi(y,h))f_{u_2+u_3}(y,h+u_1)f_{u_1}(y,h)\end{aligned}$$

Dividing both sides by

$$f_{u_1}(y,h)f_{u_3}(y,h+u_1+u_2)f_{u_2}(y,h+u_1)$$

we get

$$\begin{aligned}(\psi_{N(u_1+u_2,u_3)} \circ \pi(y,h))&\frac{f_{u_1+u_2}(y,h)}{f_{u_1}(y,h)f_{u_2}(y,h+u_1)}\\ &\sim (\psi_{N(u_1,u_2+u_3)} \circ \pi(y,h))\frac{f_{u_2+u_3}(y,h+u_1)}{f_{u_2}(y,h+u_1)f_{u_3}(y,h+u_1+u_2)}.\end{aligned}$$

Combining the above equation with equation (14),

$$\psi_{N(u_1+u_2,u_3)}\psi_{N(u_1,u_2)} = \psi_{N(u_1,u_2+u_3)}\psi_{N(u_2,u_3)}. \tag{16}$$

As $u \to f_u$ is a measurable function, $f_{u_2}(y), f_{u_2+u_1}(y)$ are close in measure for small u_1, most u_2, and the same goes for $f_{u_2}(y,h), f_{u_2}(y,h+u_1)$.

Therefore the expression in equation (14) is close (in measure) to $\bar{f_{u_1}}(y,h)$. But $N_1 \neq N_2$ implies

$$\|\psi_{N_1} - \psi_{N_2}\|_2 = \sqrt{2},$$

thus by equation (14), $\psi_{N(u_1,u_2)} = \psi_{\tilde{N}(u_1)}$ for $u_1 \in U'$ a neighborhood of zero in H, $u_2 \in A$ a set of positive measure. The set $A - A$ contains a neighborhood of zero U''. Let $U = U' \cap U''$. Take any $u_1, u_2, u_1 + u_2 \in U$, and find an element $u_3 \in A$ such that $u_3 + u_2 \in A$, then by (16)

$$\psi_{N(u_1,u_2)} = \psi_{\tilde{N}(u_1)} \psi^{-1}_{\tilde{N}(u_1+u_2)} \psi_{\tilde{N}(u_2)}$$

For $u \in U$, denote

$$\tilde{f}_u(y,h) = (\psi_{\tilde{N}(u)} \circ \pi(y,h)) f_u(y,h),$$

and

$$\tilde{\lambda}_u = \lambda_u \psi^{-1}_{\tilde{N}(u)}(\alpha).$$

By equations (14), if $u_1, u_2, u_1 + u_2 \in U$, then :

$$\tilde{f}_{u_1+u_2}(y,h) \sim \tilde{f}_{u_2}(y, h+u_1) \tilde{f}_{u_1}(y,h). \tag{17}$$

By equation(15), if $u_1, u_2, u_1 + u_2 \in U$ then

$$\tilde{\lambda}_{u_1+u_2} = \tilde{\lambda}_{u_1} \tilde{\lambda}_{u_2}. \tag{18}$$

□

3.16. **Lemma.** *Let H be a torus (possibly infinite dimensional) and let $X = Y \times_\rho H$ be an ergodic abelian extension of Y. Suppose*

$$\frac{\sigma(y,h+u)}{\sigma(y,h)} = \lambda_u \frac{f_u(T_X(y,h))}{f_u(y,h)} \tag{19}$$

for each $u \in H$, and λ_u and f_u depend measurably on u. Then there is a subgroup $J < H$ such that $H/J = \mathbb{T}^n$ such that if $\pi : H \to H/J$ is the natural projection then there exists a function $\tilde{\sigma} : Y^0 \times (H/J) \to S^1$ such that

$$\sigma(y,h) = \tilde{\sigma}(y, \pi(h)) \frac{F(T_X(y,h))}{F(y,h)}.$$

Proof. By lemma 3.15 the functions f_u can be chosen such that λ_u is multiplicative in a zero neighborhood U in H. The neighborhood U contains J_1 - a closed connected subgroup of H, such that $H/J_1 = \mathbb{T}^l$, thus λ_h is a character of J_1. Thinking of H (measurably) as $H/J_1 \times J_1$ with coordinates (h_0, j) the above equation becomes

$$\frac{\sigma(y,h_0,j+u)}{\sigma(y,h)} = \lambda_u \frac{f_u(Ty, h+\rho(y))}{f_u(y,h)}. \tag{20}$$

where $u \in J_1$. This is the same as

$$\frac{\lambda^{-1}_{j+u} \sigma(y,h_0,j+u)}{\lambda^{-1}_j \sigma(y,h)} = \frac{f_u(Ty, h+\rho(y))}{f_u(y,h)}. \tag{21}$$

Applying corollary 3.9 replacing Y with $(Y \times H/J_1)$ and H with J_1 we get

$$\lambda_j^{-1}\sigma(y, h_0, j) = \tilde{\sigma}(y, h_0)\frac{T_\rho F(y, h)}{F(y, h)}.$$

or

$$\sigma(y, h_0, j) = \lambda_j \tilde{\sigma}(y, h_0)\frac{T_\rho F(y, h)}{F(y, h)}.$$

Now for j in the kernel of λ we have $\lambda_j = 1$. The image of λ is S^1 thus if $ker\lambda$ is J then $H/J = \mathbb{T}^{l+1}$.

□

3.17. **Remark.** The group H is a compact connected abelian (metrizable) group, and therefore has countably many closed subgroups J such that H/J is a finite dimensional torus.

3.18. **Remark.** If H is any connected compact abelian group (not necessarily a torus) then J_1 in the foregoing proof is not necessarily connected. By the same proof we will get that σ is cohomologous to a cocycle lifted from a product of a finite torus and a totally disconnected compact abelian group.

3.19. **Lemma.** *Let $X = Y \times_\rho H$ be an abelian extension of Y with $\rho(y)$ cohomologous to a constant function (now H is any compact abelian group, not necessarily connected). Let σ be as in lemma 3.16. Then there is a subgroup $J < H$, and a finite group C_k, such that $H/J = \mathbb{T}^n \times C_k$ and letting $\pi : H \to H/J$ denote the natural projection then there exists a function $\tilde{\sigma} : Y^0 \times H/J \to S^1$ such that*

$$\sigma(y, h) = \tilde{\sigma}(y, \pi(h))\frac{T_X F(y, h)}{F(y, h)}.$$

Proof. By lemma 3.15 the functions f_u can be chosen such that λ_u is multiplicative in a zero neighborhood U in H. The neighborhood U contains J_1 - a closed subgroup of H, such that $H/J_1 = \mathbb{T}^l \times C_j$ where C_j is a finite group, thus λ_u is a character of J_1. Now proceed as in lemma 3.16 (the image of λ is either S^1 or a finite group). □

4. Lie Groups and Nilsystems

4.1. Let N be a k-step, simply connected nilpotent Lie group, Γ a discrete subgroup s.t. N/Γ is compact. Let $\mathcal{B}$ be the (completed) Borel algebra, m the Haar measure on N/Γ, and let $a \in N$. The system $X = (N/\Gamma, \mathcal{B}, m, T)$ where $Tg\Gamma = ag\Gamma$ is called a *nilflow*. We will sometime denote this system $(N/\Gamma, a)$. Let $N_1 = N$, and for $i > 1$: $N_i = [N_{i-1}, N]$ (N is a k-step nilpotent group if $N_{k+1} = \{1\}$), and for $i \geq 1$ let $\Gamma_i = \Gamma \cap N_i$. The groups N_j for $j > 1$ are connected (see [L02]). Th group N_k is abelian and connected and therefore isomorphic to $\mathbb{R}^m$ for some positive integer m. Then Γ_i is a

discrete subgroup of N_i, and N_i/Γ_i is compact. Let m_i be the Haar measure on N_i/Γ_i. Let

$$M = \{(y_1, y_1^2 y_2, \ldots, \prod_{j=1}^{k} y_j^{\binom{l}{j}}) : y_1 \in N_1, y_2 \in N_2, \ldots, y_k \in N_k\} \subset N^l,$$

where

$$\prod_{j=1}^{k} y_j^{\binom{l}{j}} = y_1^k y_2^{\binom{l}{2}} \ldots y_k^{\binom{l}{k}}.$$

The elements of M are called Hall-Petresco (HP) sequences, and form a group (see [La54], [L98]). The first $k+1$ elements in the sequence determine the rest. Computation shows that if $n \in N$ and $(n_1, \ldots, n_l) \in M$ then $([n, n_1], \ldots, [n, n_l]) \in M$. Let $\Lambda = M \cap \Gamma^l$. The nilmanifold $Y = M/\Lambda$ is embedded in $(N/\Gamma)^l$ and let ν be the Haar measure on Y. Then for almost all $g \in N$ for all $(n_1, \ldots n_l) \in M$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{l} f_j(a^{jn} n_j g\Gamma) = \int_Y \prod_{j=1}^{l} f_j(gz_j) d\nu(z_1, \ldots, z_l). \tag{22}$$

For more details see [Z02a]. A similar result holds for the action of $(a^{m_1}, \ldots, a^{m_l})$ for any $m_i \in \mathbb{Z}$.

4.2. **Remark.** If N/Γ is connected then a^k is ergodic for any $k \neq 0$ therefore equation 22 remains the same if we replace a by a^k.

4.3. Let $(N/\Gamma, a)$ be a k-step nilflow. Define

$$\begin{aligned} \tau_l(a) &:= a \times \ldots \times a^l \\ \triangle_l(a) &:= a \times \ldots \times a. \end{aligned}$$

Let $\triangle_l(m)$ be the diagonal measure on $(N/\Gamma)^l$.
Define a measure on $(N/\Gamma)^l$

$$\bar{\triangle}_l(m) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \tau_l(a)^n \triangle_l(m). \tag{23}$$

By the above discussion the ergodic components of $\bar{\triangle}_l(m)$ are parametrized by N/Γ and are of the form $M(g\Gamma, \ldots, g\Gamma)$.

4.4. The system $(N/\Gamma, a)$ may be represented as an Abelian extension of a $k-1$-step nilflow $(N/\Gamma, a) = (N/N_k\Gamma \times_\rho N_k/\Gamma_k)$. Inductively $(N/\Gamma, a)$ may be represented as a tower of Abelian extensions, starting out with a point. (the first block in the tower would be the Kronecker factor $N/N_2\Gamma$). Consider the system $Y = ((N/N_k\Gamma)^l, \bar{\triangle}_l)$. Then $(Y \times_{(\rho^{(1)}, \ldots, \rho^{(l)})} (N_k/\Gamma_k)^l, \tau_l(a))$ is an abelian group extension of Y. The Mackey group, associated with the

ergodic components of the group extension that are mapped onto the ergodic components of $\bar{\triangle}_l$, is

$$M_{k,l} = \{(g_1, g_1^2 g_2, \dots, \prod_{j=1}^{k} g_j^{\binom{l}{j}}) : g_1, \dots, g_k \in N_k\} \Big/ \Gamma_k^l$$

In additive notation, denote $H = N_k/\Gamma_k$, then

$$M_{k,l} = \{(h_1, 2h_1 + h_2, \dots, \sum_{j=1}^{k} \binom{l}{j} h_j) : h_1, \dots, h_j \in H\}.$$

4.5. **Lemma.** *Let $X = (N/\Gamma, a)$ be a k-step nilflow. Then $Y_r(X) = (N/N_r\Gamma, a)$ for $r \geq 2$.*

Proof. We prove this by induction on the nilpotency level of N. Let N be a 1-step nilpotent group (thus $(N/\Gamma, a)$ is a Kronecker system). Let ψ be an eigenfunction of $(N/\Gamma, a)$, then

$$\frac{1}{N} \sum_{n=1}^{N} T^n(\psi)^2 T^{2n}(\psi)^{-1} = \psi.$$

Therefore

$$L^2(Y_1) = \operatorname{span}\{\text{eigenfunctions}\} = L^2(N/\Gamma).$$

Now assume the statement for $(k-1)$-step nilflows. Let $X = (N/\Gamma, a)$ be a k-step nilflow. By lemma 2.4, for $r < k+1$, $Y_r(N/N_r\Gamma, a)$ is a factor of $Y_r(X)$. By the induction hypothesis $Y_r(N/N_r\Gamma, a) = (N/N_r\Gamma, a)$. But the integral in equation (22) for $l = r$, is a function on N/Γ that is invariant under translation by elements of N_r. By lemma 2.3, $L^2(Y_r(X))$ is spanned by these integrals. Let $r = k+1$, and let $f(g\Gamma) \in L^\infty(N/\Gamma)$. We want to show that the function f is in the span of the integrals one obtains in (22) (the value of the limit is the same for any choice of $(n_1, \dots, n_{k+1}) \in M$). The element $g\Gamma$ is determined by the first $k+1$ elements $m_1 g\Gamma, \dots, m_{k+1} g\Gamma$ of a HP geometric progression. Therefore there exists a function $F \in L^\infty \bar{\triangle}_{k+1}(m)$ such that

$$F(m_1 g\Gamma, \dots, m_{k+1} g\Gamma) = f(g\Gamma).$$

Now as

$$F(m_1 g\Gamma, \dots, m_{k+1} g\Gamma) = F(a m_1 g\Gamma, \dots, a^{k+1} m_{k+1} g\Gamma)$$

we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \tau_{k+1}(a)^n F(m_1 g\Gamma, \dots, m_{k+1} g\Gamma) = f(g\Gamma).$$

Now approximate F (in $L^2(\bar{\triangle}_{k+1}(m))$) by sums of functions of the type $f_1 \otimes \ldots \otimes f_{k+1}$. □

4.6. **Corollary.** *If Y is a k-step pro-nilflow then $Y_{k+1}(Y) = Y$.*

4.7. **Lemma.** *Let $(N/\Gamma, a)$ be a k-step nilflow. Let $f \in B(N/\Gamma, S^1)$. Let $\{\lambda_c\}$ be a family of constants, $\{f_c\}_{c\in N_k}$ be a family of functions in $B(N/\Gamma \to S^1)$ such that*

$$\frac{f(cy)}{f(y)} = \lambda_c \frac{f_c(ay)}{f_c(y)} \tag{24}$$

for all $c \in N_k$. Then we can choose f_c, λ_c such that

$$f_{c_1}(c_2 y) f_{c_2}(y) \sim f_{c_1 c_2}(y).$$

Proof. By lemma 3.15 this holds in a neighborhood of zero $U \subset N_k$. Notice that multiplying f_c by a constant does not affect equation (24). If $c \in N_k$, $c = c_1 \dots c_s$, and $c_1, \dots, c_s \in U$, define

$$f_c(y) = f_{c_1}(y) f_{c_2}(c_1 y) \dots f_{c_s}(c_1 \dots c_{s-1} y).$$

We claim this is well defined (up to a constant multiple) on N_k: given two sequences $c_1, \dots c_s$ and $c'_1, \dots c'_t$ with equal product, we can break up the 'steps' c_r into an equal number of small steps and we can interpolate a sequence of such paths where two consecutive paths differ only within a small cube which can be translated to be inside U. Since the resulting λ's and f's will be the same for consecutive paths, they will be the same for the initial and the final ones. □

4.8. **Lemma.** *Assume N/Γ has no non trivial finite factors. The group $N/N_2 \cong \mathbb{Z} \times \mathbb{R}^n$ for some integer n . The action of a on it is given by rotation by some element $(1, \alpha)$. Under the conditions of lemma 4.7 we can choose f_c, λ_c*

$$\frac{f(cy)}{f(y)} = e^{2\pi i \langle L\alpha, c\rangle} \frac{f_c(ay)}{f_c(y)},$$

for an integer $n \times m$ integer matrix L.

Proof. Now λ_c is a continuous multiplicative function on $N_k = \mathbb{R}^m$. We now use additive notation for N_k. In this notation λ_c is of the form $e^{2\pi i\langle r,c\rangle}$. Let e_i denote the standard basis for $\mathbb{R}^m$. Each f_{e_i} is an eigenfunction (as the left side of equation (24) is 1). Thus there exists $\vec{n}_i \in \mathbb{Z}^n$ such that

$$f_{e_i}(y) = Ce^{2\pi i\langle \vec{n}_i, \bar{y}\rangle},$$

where $\bar{y} \in N/N_2$, with eigenvalue $e^{2\pi i\langle \vec{n}_i, \alpha\rangle}$. Finally for each i there is $k_i \in \mathbb{Z}$ such that

$$\langle r, e_i\rangle = \langle \vec{n}_i, \alpha\rangle + k_i.$$

Now take L the matrix with the i'th row being $(k_i, \vec{n}_i)$ (the action of a on $\mathbb{Z} \times \mathbb{R}^n$ given by $(1, \alpha)$). □

4.9. **Lemma.** *For any $c_1, c_2 \in N_k$, f_{c_1} and f_{c_2} satisfying equation (24) we have*

$$\frac{f_{c_1}(c_2 y)}{f_{c_1}(y)} = \frac{f_{c_2}(c_1 y)}{f_{c_2}(y)}$$

Proof. The function

$$\frac{f_{c_1}(c_2y)}{f_{c_1}(y)} \Big/ \frac{f_{c_2}(c_1y)}{f_{c_2}(y)}$$

is invariant under rotation by a, therefore

$$\frac{f_{c_1}(c_2y)}{f_{c_1}(y)} = C(c_1,c_2)\frac{f_{c_2}(c_1y)}{f_{c_2}(y)}$$

Using lemma 4.7

$$\begin{aligned}C(c_1,c_2c)\frac{f_{c_2}(c_1cy)f_c(c_1y)}{f_{c_2}(cy)f_c(y)} &= C(c_1,c_2c)\frac{f_{c_2c}(c_1y)}{f_{c_2c}(y)} = \frac{f_{c_1}(c_2cy)}{f_{c_1}(y)}\\ =&\frac{f_{c_1}(c_2cy)}{f_{c_1}(cy)}\frac{f_{c_1}(cy)}{f_{c_1}(y)} = C(c_1,c_2)C(c_1,c)\frac{f_{c_2}(c_1cy)}{f_{c_2}(y)}\frac{f_c(c_1y)}{f_c(y)}\end{aligned}$$

Therefore $C(c_1,c_2)$ is multiplicative in c_1, c_2. If $c \in N_k \cap \Gamma$ then f_c is an eigenfunction. Thus for $c \in N_k \cap \Gamma$, $C(c_1,c) = C(c,c_2){=}1$. This implies that for any $c_1 \in N_k$, $C(c_1,*)$ is a character of $N_k/(N_k \cap \Gamma)$ which is a compact connected abelian group. As there are countably many of those $C(c_1,c_2) \equiv 1$. □

5. The van der Corput lemma

One of the main tools in studying characteristic factors is the van der Corput lemma. The formulation below is due to Bergelson [Be87]:

5.1. **Lemma** (van der Corput). *Let $\{u_n\}$ be a bounded sequence of vectors in a Hilbert space $\mathcal{H}$. Assume that for each m the limit*

$$\gamma_m := \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle u_n, u_{n+m}\rangle$$

exists, and

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\gamma_m = 0. \tag{25}$$

Then

$$\frac{1}{N}\sum_{n=1}^{N}u_n \xrightarrow{\mathcal{H}} 0.$$

Proof. Let M be large enough so that the expression in (25) is small. Let N be large enough with respect to M so that the two expressions

$$\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{n+m}, \quad \frac{1}{N}\sum_{n=1}^{N}u_n$$

are close. We have:

$$\|\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{n+m}\|^2 \leq \frac{1}{N}\sum_{n=1}^{N}\|\frac{1}{M}\sum_{m=1}^{M}u_{n+m}\|^2$$
$$= \frac{1}{NM^2}\sum_{n=1}^{N}\sum_{m_1,m_2=1}^{M}\langle u_{n+m_1},u_{n+m_2}\rangle$$
$$\overset{N\to\infty}{\longrightarrow} \frac{1}{M^2}\sum_{m_1,m_2=1}^{M}\gamma_{m_2-m_1}$$

which is small. □

6. proof of Theorem 1.7

Let X be an ergodic m.p.s, and let $Y_j(X)$ be the j-u.c.f of X, and let $\pi_j : X \to Y_j(X)$ be the factor map. When the context is clear we will write T for T_X, and Y_j for $Y_j(X)$. Let $\vec{a} = (a_1, \dots, a_l) \in \mathbb{Z}^l$. We will always assume that a_i are distinct. Denote

$$\tau_{\vec{a}}(T) := T^{a_1} \times \ldots \times T^{a_l}$$
$$\triangle_l(T) := T \times \ldots \times T.$$

When the context is clear we will use $\tau_{\vec{a}}$ for $\tau_{\vec{a}}(T)$, and T or T_l for $\triangle_l(T)$.

Let $\triangle_l(\mu_X)$ be the diagonal measure on $(X^0)^l$. We will prove theorem 1.7 inductively, along with a sequence of statements (theorem 1.7 is item (7)).

6.1. **Theorem.** (1) *Let $\vec{a} = (a_1, \dots, a_{j+1}) \in \mathbb{Z}^{j+1}$. The limit*

$$\bar{\triangle}_{\vec{a}}(\mu_X) := \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tau_{\vec{a}}^n\triangle_{j+1}(\mu_X).$$

exists. Furthermore $\bar{\triangle}_{\vec{a}}(\mu_X)$ is the conditional product measure relative to $\bar{\triangle}_{\vec{a}}(\mu_{Y_j})$.

(2) *$Y_{j+1}(X)$ is an isometric extension of $Y_j(X)$.*

(3) *Let X be an ergodic m.p.s. Let $l \in \mathbb{N}$. Let μ be a measure on $(X^0)^l$, let $\vec{a} = (a_1, \dots, a_l) \in \mathbb{Z}^l$, and for $i = 1, \dots, l$, let $f_i \in B(X^0, S^1)$. Recall that $f^{(m)}(x) = f(T^{m-1}x)\dots f(Tx)f(x.)$ We say that $(f_1, f_2, \dots, f_l)$ is of* type *$\vec{a}$ w.r.t μ if there exists a μ-measurable function F taking values in S^1, such that*

$$\prod_{i=1}^{l} f_i^{(a_i)}(x_i) = \frac{\tau_{\vec{a}}F(x_1,\dots,x_l)}{F(x_1,\dots,x_l)}.$$

Let H be a compact abelian group, Y a $(j-1)$-step pro-nilflow. We say that $\rho : Y \to H$ is of type *j if for any character $\chi \in \hat{H}$, there exists a character $\tilde{\chi} = (\chi_1, \dots, \chi_l) \in \hat{H}^l$, and integers $\vec{a} \in \mathbb{Z}^l$, such that $\chi = \chi_k$ for some $l \geq k \geq 1$ and $(\chi_1 \circ \rho, \dots, \chi_l \circ \rho)$ is of type $\vec{a}$ w.r.t*

. $\bar{\triangle}_{\vec{a}}(\mu_Y)$. Let Y be a $(j-1)$-step pro-nilflow, and let $(f_1, f_2, \dots, f_l)$ be of type $\vec{a}$ w.r.t $\bar{\triangle}_{\vec{a}}(\mu_Y)$. Then

(a) *f_k is cohomologous to a function lifted from a $(j-1)$-step nilflow.*
(b) *For $k = 1, \dots, l$, f_k belongs to a countable set modulo coboundaries.*
(c) *For $k = 1, \dots, l$, $f_k : Y \to S^1$ is of type j.*
(d) *If $\rho : Y \to H$ is of type j then for any character χ of H, $\chi \circ \rho$ is of type j.*
(e) *If $f, g : Y \to S^1$ are of type j, then fg is of type j.*

(4) *If $X = Y_j(X) \times_\sigma H$ is an abelian extension by a cocycle of type j, then X can be given the structure of a j-step pro-nilflow. If $Y_j(X)$ is a nilflow and H is a finite dimensional torus, then X is a nilflow.*
(5) *A factor of a j-step pro-nilflow is a j-step pro-nilflow.*
(6) *If X a j-step pro-nilflow then $X = Y_j(X) \times_\sigma H$ is an abelian extension of $Y_j(X)$ by a cocycle of type j. If $j \geq 1$ then H is connected.*
(7) *$Y_{j+1}(X)$ can be given a structure of a j-step pro-nilflow.*
(8) *Let $a_1, \dots a_{j+1} \in \mathbb{Z}$, and $f_1 \dots f_{j+1} \in L^\infty(\mu_X)$. Then the averages*

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{k=1}^{j+1} f_k(T^{a_k n} x)$$

converge in $L^2(\mu_X)$.

Proof. For $j = 0$, $Y_{j+1}(X)$ is the pro-cyclic factor. For $j = 1$, $Y_{j+1}(X)$ is the Kronecker factor which is an abelian extension (by a connected group) of the pro-cyclic factor by a constant cocycle, and all statements are easily verified. Assume all statements hold replacing j with $j-1$.

6.2. *proof of theorem 6.1 (1)*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int \tau_{\vec{a}}^n f_1 \otimes \dots \otimes f_{j+1} d\triangle_{j+1}(\mu_X) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f_1(T^{na_1}x) f_2(T^{na_2}x) \dots f_{j+1}(T^{na_{j+1}}x) d\mu_X \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f_1(x) f_2(T^{n(a_2-a_1)}x) \dots f_{j+1}(T^{n(a_{j+1}-a_1)}x) d\mu_X \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int E(f_1|Y_j)(\pi_j x) \prod_{i=1}^{j} (T^{n(a_{i+1}-a_1)} E(f_{i+1}|Y_j)(\pi_j x)) d\mu_{Y_j} \\
&= \int E(f_1|Y_j) \otimes \dots \otimes E(f_{j+1}|Y_j) d\bar{\triangle}_{\vec{a}}(\mu_{Y_j}).
\end{aligned}$$

By the above calculation

$$\int f_1\otimes\ldots\otimes f_{j+1} d\bar{\triangle}_{\vec{a}}(\mu_X). = \int E(f_1|Y_j)\otimes\ldots\otimes E(f_{j+1}|Y_j)d\bar{\triangle}_{\vec{a}}(\mu_{Y_j}), \tag{26}$$

thus $\bar{\triangle}_{\vec{a}}(\mu_X)$ is the conditional product measure relative to $\bar{\triangle}_{\vec{a}}(\mu_{Y_j})$.

6.3. *proof of theorem 6.1(2).*

We must show that if $E(f_k|\mathcal{B}_{\hat{Y}_j(X)}) = 0$ for some k, then the averages $\frac{1}{N}\sum_{n=1}^N\prod_{k=1}^{j+1} f_k(T^{a_k n}x)$ converge to zero in $L^2(\mu_X)$. We apply the van der Corput Lemma 5.1 with

$$u_n = \prod_{k=1}^{j+1} T^{na_k} f_k(x).$$

We calculate γ_m:

$$\begin{aligned}
\gamma_m &= \lim \frac{1}{N}\sum_{n=1}^N \langle u_n, u_{n+m}\rangle \\
&= \lim \frac{1}{N}\sum_{n=1}^N \int \prod_{k=1}^{j+1} T^{na_k} f_k(x) T^{na_k+ma_k} f_k(x) d\mu_X \\
&= \lim \frac{1}{N}\sum_{n=1}^N \int \prod_{k=1}^{j+1} T^{na_k}(f_k T^{ma_k} f_k(x)) d\mu_X \\
&= \int (f_1\otimes\ldots\otimes f_{j+1})\,\tau_{\vec{a}}^m\,(f_1\otimes\ldots\otimes f_{j+1})\,d\bar{\triangle}_{\vec{a}}(\mu_X).
\end{aligned}$$

By the ergodic theorem, there exists a $\tau_{\vec{a}}$ invariant function $D_{\vec{a}} \in L^2(\bar{\triangle}_{\vec{a}}(\mu_X))$ such that

$$\lim \frac{1}{M}\sum_{m=1}^M \gamma_m = \int f_1\otimes\ldots\otimes f_{j+1} D_{\vec{a}}(x_1,\ldots,x_{j+1})d\bar{\triangle}_{\vec{a}}(\mu_X). \tag{27}$$

By 6.2, $\bar{\triangle}_{\vec{a}}(\mu_X)$ is the conditional product measure relative to $\bar{\triangle}_{\vec{a}}(\mu_{Y_j})$. By theorem 2.11, $D_{\vec{a}}$ is measurable w.r.t $(\hat{Y}_j(X))^{j+1}$. If $E(f_k|\mathcal{B}_{\hat{Y}_j(X)}) = 0$ then the average (27) is zero, and by VDC so is the original average.

6.4. *proof of theorem 6.1(3).*

This part is the bulk of the theorem, and the proof of its items will be intertwined with the proof of the rest of the items in theorem 6.1. Let Y a $(j-1)$-step pro-nilflow, with $j \geq 2$. By corollary 4.6, $Y = Y_j(Y)$. By the induction hypothesis in theorem 6.1(6), we can identify Y with a presentation as a tower of abelian extensions $Y = H_1 \times_{\sigma_1} \times H_2 \times \ldots \times_{\sigma_{j-1}} H_j$ where σ_i is of type i, H_i is connected for $i > 1$, and $Y_i(Y) = H_1 \times_{\sigma_1} \times H_2 \times \ldots \times_{\sigma_{i-1}} H_i$. Specifically $Y = Y_{j-1}(Y) \times_{\sigma_{j-1}} H_j$, where H_j is a connected compact abelian group, and σ_{j-1} is of type $j-1$. Denote $Y_{j-1} = Y_{j-1}(Y)$. Let $\pi_{j-1} : Y \to Y_{j-1}$ be the projection. We identify $y \in Y$ with $(\pi_{j-1}y, h) \in$

$\pi_{j-1}Y \times H_j$. Let l be a positive integer. To simplify the notation we now restrict ourselves to the special case where $\vec{a} = (1, 2, \ldots, l)$. The analysis is similar for any $\vec{a} \subset \mathbb{Z}^l$.

We write $\bar{\triangle}_l(\mu_Y)$ for the measure $\bar{\triangle}_{(1,\ldots,l)}(\mu_Y)$, and we say that $(f_1, \ldots, f_l)$ is of type l w.r.t $\bar{\triangle}_l(\mu_Y)$ if $(f_1, \ldots, f_l)$ is of type $(1, \ldots, l)$ w.r.t $\bar{\triangle}_l(\mu_Y)$. In this case there exists a function $F \in L^\infty(\bar{\triangle}_l(\mu_Y))$ such that

$$\prod_{k=1}^{l} f_k^{(k)}(y_k) = \frac{\tau F(y_1 \ldots, y_l)}{F(y_1 \ldots, y_l)}. \tag{28}$$

6.5. **Remark.** As Y is a $(j-1)$-step pro-nilflow, on the support of $\mu_l(Y)$, the coordinates $y_{j+1}, \ldots, y_l$ are determined by the first j coordinates $y_1, \ldots, y_j$, and this correspondence is invariant under τ (if $j = 2$ then Y is an abelian group, and y_1, y_2, y_3 form an arithmetic sequence, in general see the discussion in 4.1). Therefore the function $F(y_1, \ldots, y_l)$ can be replaced by a function of j coordinates, and equation (28) can be written in the form

$$\prod_{k=1}^{l} f_k^{(k)}(y_k) = \frac{\tau F(y_1, \ldots, y_j)}{F(y_1, \ldots, y_j)}. \tag{29}$$

We will repeatedly refer to this equation.

6.6. **Remark.** The measure $\bar{\triangle}_j(\mu_Y) = \bar{\triangle}_j(\mu_{Y_{j-1}}) \times (m_{H_j})^j$ (replace X by Y and j by $j-1$ in equation (26)).

6.7. **Lemma.** *Let $(f_1, \ldots, f_{j+1})$ be of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_Y)$. Then for each $k = 1, \ldots, j+1$ there exists a family of functions $\{g_{k,u}\}_{u \in H_j} \subset B(Y^0, S^1)$, and a family of functions $\{f_{k,u}\}_{u \in H_j} \subset B(Y^0, S^1)$ such that*

$$\frac{f_k(\pi_{j-1}y, h+u)}{f_k(\pi_{j-1}y, h)} = g_{k,u}(\pi_{j-1}y)\frac{Tf_{k,u}(y)}{f_{k,u}(y)}. \tag{30}$$

Proof. We use the fact that $\bar{\triangle}_{j+1}(Y)$ is invariant under translations by elements of the Mackey group $M_{j-1,j+1}$, and by elements of $\triangle_{j+1}(H_j) = \{(h, \ldots, h)\}_{h \in H_j} \subset H_j^{j+1})$. This group is described in 4.4. Let

$$M_{j-1,j+1}(0) = (M_{j-1,j+1} + \triangle_{j+1}(H_j)) \cap (H_j^j \times \{0\})$$

The projection of $M_{j-1,j+1}(0)$ on any j coordinates is full (i.e H_j^j). Let $\vec{u} = (u_1, \ldots, u_{j+1}) \in M_{j-1,j+1}(0)$ $(u_{j+1} = 0)$. Then

$$\prod_{k=1}^{j+1} f_k^{(k)}(\pi_{j-1}y_k, h_k + u_k) = \frac{\tau F(\pi_{j-1}y_1, h_1 + u_1, \ldots, \pi_{j-1}y_j, h_j + u_j)}{F(\pi_{j-1}y_1, h_1 + u_1, \ldots, \pi_{j-1}y_j, h_j + u_j)}. \tag{31}$$

Dividing equation (31) by equation (29) we get

$$\prod_{k=1}^{j} \frac{f_k^{(k)}(\pi_{j-1}y_k, h_k + u_k)}{f_k^{(k)}(\pi_{j-1}y_k, h_k)} = \frac{\tau F_{\vec{u}}(y_1, \ldots, y_j)}{F_{\vec{u}}(y_1, \ldots, y_j)}, \tag{32}$$

where

$$F_{\vec{u}}(y_1,\dots,y_j)=\frac{F(\pi_{j-1}y_1,h_1+u_1,\dots,\pi_{j-1}y_j,h_j+u_j)}{F(y_1,\dots,y_{j+1})}$$

Using remark 6.6 we can apply theorem 3.8 to $\bar{\triangle}_j(\mu_Y)$. For any $1\le k\le j$ and any $u_k\in H_j$ there exist functions $g_{k,u_k}\in B(Y^0,S^1)$ and $f_{k,u_k}\in B(Y^0,S^1)$ such that

$$\frac{f_k^{(k)}(\pi_{j-1}y,h+u_k)}{f_k^{(k)}(\pi_{j-1}y,h)}=g_{k,u_k}(\pi_{j-1}y)\frac{T^kf_{k,u_k}(y)}{f_{k,u_k}(y)}. \tag{33}$$

We still need to show that the same holds for f_k (rather than $f_k^{(k)}$): we use the fact that T_{f_k} (see 2.7) and $T^k_{f_k^{(k)}}=(T_{f_k})^k$ commute therefore

$$f_k^{(k)}(Ty)f_k(y)=f_k(T^ky)f_k^{(k)}(y). \tag{34}$$

Using equations (33),(34), and a calculation we find that the function

$$\frac{f_k(\pi_{j-1}y,h+u+v)/f_k(\pi_{j-1}y,h+u)}{f_k(\pi_{j-1}y,h+v)/f_k(\pi_{j-1}y,h)}\bigg/\frac{T(f_{k,v}(\pi_{j-1}y,h+u)/f_{k,v}(\pi_{j-1}y,h))}{f_{k,v}(\pi_{j-1}y,h+u)/f_{k,v}(\pi_{j-1}y,h)}$$

is T^k invariant and therefore constant on the (finitely many) ergodic components of T^k. Denote this constant $\delta_{k,u,v}(\pi_1y)$ (any ergodic component of T^k is determined by a cyclic group which is a factor of $Y_1(X)$ same argument as in 3.8). By lemma 3.15, $\delta_{k,u,v}(\pi_1y)$ is multiplicative in u (also in v) in a neighborhood of zero in H_j. By equation (33) (after iteration) $(\delta_{k,u,v})^k(\pi_1y)$ is an eigenvalue, therefore $\delta_{k,u,v}(\pi_1y)=1$ for u in a neighborhood of zero in H_j. Iterating we find this is true for all $u\in H_j$ (H_j is connected). By corollary 3.9

$$\frac{f_k(\pi_{j-1}y,h+v)}{f_k(\pi_{j-1}y,h)}=\tilde{g}_{k,v}(\pi_{j-1}y)\frac{T\tilde{f}_{k,v}(\pi_{j-1}y,h)}{\tilde{f}_{k,v}(\pi_{j-1}y,h)}.$$

□

6.8. **Lemma.** *If $(g_1\circ\pi_{j-1},\dots,g_l\circ\pi_{j-1})$ is of type l w.r.t $\bar{\triangle}_j(\mu_Y)$, then as a function on $\pi_{j-1}Y$, g_k is of type $j-1$, for $k=1,\dots,l$.*

Proof. By definition there exists a $\bar{\triangle}_l(\mu_Y)$ measurable function L such that

$$\prod_{k=1}^{l}(g_k)^{(k)}(\pi_{j-1}y_k)=\frac{\tau L(y_1,\dots,y_l)}{L(y_1,\dots,y_l)}.$$

Taking the Fourier expansion of L with respect to the abelian group H_j^j

$$L(y_1,\dots,y_l)=\sum_{\chi\in\hat{H}_j^j}G_\chi(\pi_{j-1}y_1,\dots,\pi_{j-1}y_l)\chi_1(h_1)\dots\chi_j(h_l),$$

We find that for any $\chi \in \hat{H}_j^l$

$$G_\chi(y_1,\dots,y_l)\prod_{k=1}^{l} g_k^{(k)}(\pi_{j-1}y_k)\bar{\chi}_k(\sigma_{j-1}^{(k)}(\pi_{j-1}y_k)) = \tau G_\chi(\pi_{j-1}y_1,\dots,\pi_{j-1}y_l).$$

The function $|G_\chi|$ is invariant under τ and therefore constant on the ergodic components of $\bar{\triangle}_l(\mu_{Y_{j-1}})$. As $\hat{H}_j^l$ is countable, there is a character $\chi \in \hat{H}_j^l$, and a set of $\bar{\triangle}_l(\mu_{Y_{j-1}})$ positive measure A, that is τ invariant, and for which

$$\prod_{k=1}^{l} g_k^{(k)}(\pi_{j-1}y_k)\bar{\chi}_k\sigma_{j-1}^{(k)}(\pi_{j-1}y_k) = \frac{\tau G_\chi(\pi_{j-1}y_1,\dots,\pi_{j-1}y_l)}{G_\chi(\pi_{j-1}y_1,\dots,\pi_{j-1}y_l)}. \tag{35}$$

Denote by W the system $(Y_{j-1}^l, \bar{\triangle}_l(\mu_{Y_{j-1}}), \tau)$. Let $\rho_k = g_k\bar{\chi}_k(\sigma_{j-1}) : Y_{j-1} \to S^1$, and let $\tilde{\rho} = (\rho_1, \rho_2^{(2)}, \dots, \rho_l^{(l)})$. Denote $\bar{y} := \pi_{j-1}y$. Consider the group extension $W \times_{\tilde{\rho}} (S^1)^l$. Let $W_{\bar{y}}$ be an ergodic component of W (the ergodic components of W are parametrized by $\bar{y} \in Y_{j-1}$), and let $P_{\bar{y}} \subset (S^1)^l$ be the Mackey group associated with the ergodic components of the group extension $W_{\bar{y}} \times_{\tilde{\rho}} (S^1)^l$. As the transformations

$$T_{\rho_1} \times (T_{\rho_2})^2 \times \dots \times (T_{\rho_l})^l, \quad T_{\rho_1} \times T_{\rho_2} \times \dots \times T_{\rho_l}$$

commute, $P_{T\bar{y}} = P_{\bar{y}}$ (see lemma 2.18), and by ergodicity $P_{\bar{y}} = P$ is constant a.e. as a function of $\bar{y}$. As equation (35) holds on a τ invariant set of $\bar{\triangle}_l(\mu_{Y_{j-1}})$ positive measure, we get

$$(\varphi,\dots,\varphi) \in P^\perp,$$

where $\varphi(\zeta) = \zeta$. Therefore for a.e. $\bar{y}$, there exists a function $G_{\bar{y}}$, so that equation (35) holds on $W_{\bar{y}}$ replacing G_χ with $G_{\bar{y}}$. Notice that $G_{\bar{y}}$ is determined up to a constant multiple on $W_{\bar{y}}$, and can therefore be chosen so that it depends measurably on $\bar{y}$ (see 3.13). Thus there exists a measurable function G such that equation (35) holds $\bar{\triangle}_l(\mu_{Y_{j-1}})$ a.e. replacing G_χ by G. This implies that $(g_1 \cdot \bar{\chi}_1(\sigma_{j-1}), \dots, g_l \cdot \bar{\chi}_l(\sigma_{j-1}))$ is of type l w.r.t. $\bar{\triangle}_l(\mu_{Y_{j-1}})$. By the induction hypothesis in 6.1(3c), (3d) the functions $g_k\chi_k(\sigma_{j-1})$, $\chi_k(\sigma_{j-1})$ are of type $j-1$. By the induction hypothesis in 6.1(3e), g_k is of type $j-1$. □

6.9. **Corollary.** *Let $(f_1,\dots,f_{j+1})$ be of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_Y)$. Let $\vec{u} \in M_{j-1,j+1}(0)$, and let g_{k,u_k} satisfy equation (30). Then for $k = 1,\dots j+1$, as a function on $\pi_{j-1}Y$, g_{k,u_k} is of type $j-1$.*

Proof. Substitute equation (33) in equation (32) and use lemma 6.8. □

6.10. **Corollary.** *If $(f_1,\dots f_{j+1})$ is of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_Y)$ then for $k = 1,\dots,j+1$ there exists a family of constants $\{\lambda_{k,u}\}_{u\in H_j}$, and a family of measurable functions $\{f_{k,u}\}_{u\in H_j}$ such that*

$$\frac{f_k(\pi_{j-1}y, h+u)}{f_k(\pi_{j-1}y, h)} = \lambda_{k,u}\frac{Tf_{k,u}(y)}{f_{k,u}(y)}. \tag{36}$$

Proof. By theorem 6.1(3b), using the induction hypothesis, there are countably many $g_{k,u}$ up to $\pi_{j-1}Y$-quasi-coboundaries. There exists a set U of positive measure in H_j such that

$$u, v \in U \Rightarrow \frac{g_{k,u}}{g_{k,v}} \text{ is a quasi-coboundary.}$$

If $u, u+v \in U$ and if $f_{k,u,v} = f_{k,u+v}/f_{k,u}$ then

$$\frac{f_k(\pi_{j-1}y, h+u+v)}{f_k(\pi_{j-1}y, h+u)} = \frac{f_k(\pi_{j-1}y, h+u+v)/f_k(\pi_{j-1}y, h)}{f_k(\pi_{j-1}y, h+u)/f_k(\pi_{j-1}y, h)} = C_{u,v,k}\frac{Tf_{k,u,v}(y)}{f_{k,u,v}(y)}.$$

Thus the claim is true for v in a neighborhood of zero in H_j (as the map $H \to H$ sending h to $h+u$ is onto). As H_j is connected equation (36) holds for all $v \in H_j$. □

6.11. **Lemma.** *The families in the previous lemma can be chosen so that the function $\lambda_{k,u} : H_j \to S^1$ is multiplicative in a neighborhood of zero in H_j.*

Proof. By lemma 3.15. □

6.12. **Corollary.** *The functions $f_{k,u}$ can be chosen so that for some neighborhood of zero $U \subset H$, for any $\vec{u} = (u_1, \ldots, u_{j+1}) \in M_{j-1,j+1} + \triangle_{j+1}(H_j) \cap U^{j+1}$*

$$\prod_{k=1}^{j+1} \lambda_{k,u_k}^k = 1$$

Proof. Choose the families $\{f_{k,u}\}, \{\lambda_{k,u}\}$ so that $\lambda_{k,u}$ is multiplicative in a a neighborhood of zero in H_j. Substituting equation (36) in equation (32), we find that $\prod_{k=1}^{j+1} \lambda_{k,u_k}^k$ is an eigenvalue of τ. □

6.13. **Lemma.** *Let $(f_1, \ldots, f_{j+1})$ be of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_Y)$. Then there exists an integer n, and a factor $\tilde{Y} = Y_{j-1} \times \mathbb{T}^n$ of Y, such that if $p : Y \to \tilde{Y}$ is the factor map, then for $k = 1, \ldots, j+1$, f_k is cohomologous to a cocycle $\tilde{f}_k \circ p$. Furthermore, there exist functions $g_1, \ldots, g_{j+1}$, where $g_k : Y_{j-1}^0 \to S^1$ is of type $j-1$, such that*

$$(\tilde{f}_1 g_1 \circ \pi_{j-1}, \ldots, \tilde{f}_{j+1} g_{j+1} \circ \pi_{j-1})$$

is of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_{\tilde{Y}})$, and therefore corollary 6.10 holds replacing f_k by $\tilde{f}_k$, and Y by $\tilde{Y}$ (π_{j-1} is the projection $\tilde{Y} \to Y_{j-1}$).

Proof. By lemma 3.16, and remark 3.18 we can find J so that $H_j/J = \mathbb{T}^n \times \tilde{H}$ where $\tilde{H}$ is a compact totally disconnected abelian group (lemma 3.16 can be carried out simultaneously for $(f_1, \ldots, f_{j+1})$). Consider the system $W = Y_{j-1} \times_{\sigma_{j-1}} (\mathbb{T}^n \times \tilde{H})$. This system is a factor of Y_j, therefore $Y_{j-1}(W) = Y_{j-1}$, W is a $(j-1)$-step pro-nilflow, and $Y_j(W) = W$ (this follows from the induction hypothesis in theorem 6.1(5), and from corollary 4.6). This a contradiction to $Y_j(W)$ being an extension of $Y_{j-1}(W)$ by a

connected abelian group for $j > 1$ (this follows from the induction hypothesis in 6.1(6)). Now Y can be presented as a skew product $Y = \tilde{Y} \times_{\tilde{\sigma}_{j-1}} J$ where

$$\tilde{\sigma}_{j-1}(\tilde{y}) = \bar{r}(\tilde{y})\sigma_{j-1}(\pi_{j-1}(\tilde{y}))r(T\tilde{y}),$$

where $r(\tilde{y})$ takes values in H_j. As in the proof of lemma 6.8, there exists a character $(\chi_1, \dots, \chi_{j+1}) \in \hat{J}^{j+1}$ such that

$$(\tilde{f}_1 \cdot \chi_1(\tilde{\sigma}_{j-1}), \dots, \tilde{f}_{j+1} \cdot \chi_{j+1}(\tilde{\sigma}_{j-1}))$$

is of type $j+1$ w.r.t. $\bar{\triangle}_{j+1}(\mu_{\tilde{Y}})$. Let $\tilde{\chi}_k$ be the lift of χ_k to a character of H_j, then

$$\tilde{\chi}_k(\tilde{\sigma}_{j-1}(\tilde{y})) = \tilde{\chi}_k(\bar{r}(\tilde{y}))\tilde{\chi}_k(\sigma_{j-1}(\pi_{j-1}(\tilde{y})))\tilde{\chi}_k(r(T\tilde{y}).$$

Therefore

$$(\tilde{f}_1 \cdot \tilde{\chi}_1(\sigma_{j-1}(\pi_{j-1})), \dots, \tilde{f}_{j+1} \cdot \tilde{\chi}_{j+1}(\sigma_{j-1}(\pi_{j-1}))$$

is of type $j+1$ w.r.t. $\bar{\triangle}_{j+1}(\mu_{\tilde{Y}})$. $\square$

6.14**. Notation.** If U is an abelian group we denote rotation by an element of U by R_u.

6.15**. Lemma.** *Let $(f_1, \dots, f_{j+1})$ be of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_Y)$. Then there exists a factor of Y which is a $(j-1)$-step nilflow $\tilde{Y} = (N/\Gamma, a)$, such that if $p : Y \to \tilde{Y}$ is the factor map, then for $k = 1, \dots j+1$*

1. *f_k is cohomologous to $\tilde{f}_k \circ p$.*
2. *There exist functions $g_1, \dots, g_{j+1}$, where $g_k : \tilde{Y}^0 \to S^1$ is of type $j-1$, such that $(\tilde{f}_1 g_1 \circ \pi_{j-1}, \dots, \tilde{f}_{j+1} g_{j+1} \circ \pi_{j-1})$ is of type $j+1$ w.r.t $\bar{\triangle}_{j+1}(\mu_{\tilde{Y}})$ (π_{j-1} is the projection $\tilde{Y} \to \pi_{j-1}(\tilde{Y})$)*
3. *Corollary 6.10 holds replacing f_k by $\tilde{f}_k$, and Y by $\tilde{Y}$.*

Proof. Recall the identification of Y as a tower of abelian extensions $Y = H_1 \times_{\sigma_1} H_2 \times \dots \times_{\sigma_{j-1}} H_j$, where $Y_i(Y) = Y_{i-1}(Y) \times_{\sigma_{i-1}} H_i$, the group H_i is an abelian group which is connected for $i > 1$, and σ_i is of type i . We would like to 'replace' H_i for $i \geq 2$ with a finite dimensional torus $\mathbb{T}^{n_i}$ (and replace H_1 by a cyclic group) and get a system $C \times_{\sigma'_1} \mathbb{T}^{n_2} \times_{\sigma'_2} \dots \times_{\sigma'_{j-1}} \mathbb{T}^{n_j}$ that is a factor of Y and therefore a nilflow (a pro-nilflow by the induction hypothesis in theorem 6.1(5), and a nilflow by the induction hypothesis in 6.1(4)), and if p is the factor map, then f_k is cohomologous to $\tilde{f}_k \circ p$.

We do this by decreasing induction on the index i. The case $i = j$ was proved in lemma 6.13. Assume we have constructed a system $\tilde{Y} = H_1 \times_{\sigma_1} \dots \times H_i \times_{\sigma'_i} \mathbb{T}^{n_{i+1}} \times \dots \times_{\sigma'_{j-1}} \mathbb{T}^{n_j}$ that is a factor of Y and satisfies $(1)-(3)$. We may now forget the original system Y. By abuse of notation we replace $\tilde{Y}$ by Y, and $\tilde{f}_k$ by f_k. The cocycles $\sigma'_i, \dots, \sigma'_{j-1}, g_1, \dots g_{j+1}$ are of type $< j$ and take values in finite dimensional tori. Therefore there exists a finite dimensional torus $\mathbb{T}^{n_i}$ with σ'_l for $l = i, \dots, j-1$, cohomologous to functions $\tilde{\sigma}_l$, lifted from $Y'_l := H_1 \times \dots \times H_{i-1} \times_{\sigma'_{i-1}} \mathbb{T}^{n_i} \times \dots \times_{\sigma'_{l-1}} \mathbb{T}^{n_l}$, and g_k, for $k = 1, \dots j+1$ cohomologous to $\tilde{g}_k$ lifted from $Y' := Y'_{j+1}$.

After reparametrization we may assume σ'_l, for $l = i, \dots, j-1$, is lifted from Y'_l. Condition (2) remains valid if we can replace g_k by $\tilde{g}_k$. Write $H_i = \mathbb{T}^{n_i} \times U$ (measurewise) where U is a compact abelian group. The action of T_Y commutes with rotation by an element in U; i.e with

$$R_{u'} : (h_1, \dots, h_{i-1}, t_i, u, t_{i+1}, \dots, t_j) \to (h_1, \dots, h_{i-1}, t_i, u + u', t_{i+1}, \dots, t_j)$$

Indeed, $\sigma'_i, \dots, \sigma'_{j-1}$ are not affected by translation in elements of U. Both R_u, T_Y commute with rotation by an element $t \in \mathbb{T}^{n_j}$. Therefore

$$\frac{f_k(R_u(\pi_{j-1}y, t_j + t))}{f_k(R_u(\pi_{j-1}y, h))} \Big/ \frac{f_k(\pi_{j-1}y, t_j + t)}{f_k(\pi_{j-1}y, t_j)} = \frac{f_{k,v}(R_u T y)}{f_{k,t}(Ty)} \Big/ \frac{f_{k,t}(R_u y)}{f_{k,t}(y)} .$$

By Theorem 3.6

$$f_k(R_u y)/f_k(y)$$

is cohomologous to a cocycle lifted from $\pi_{j-1}Y$. By the same argument as in lemma 6.10 it is cohomologous to a constant for u in a neighborhood of zero in U_i.

Now proceed as in lemma 6.13 to obtain $\tilde{Y}$ (which will be a factor between Y' and Y). Applying the same procedure for $i = 1$ gives the cyclic part. □

6.16. **Remark.** Let $f_1, \dots, f_{j+1}$ be as in lemma 6.15. By remark 3.17 there are countably many possibilities for the groups U_i and therefore countably many possibilities for the nilflow N/Γ (up to isomorphism).

6.17. Let $(N/\Gamma, a)$ be the $(j-1)$-step nilflow from lemma 6.15, and let $p : Y \to N/\Gamma$ be the projection. Let $N_1 = N$, and $N_{l+1} = [N, N_l]$ ($N_j = \{1\}$). We will show that if the system $(N/\Gamma, a)$ has no finite non trivial factors, then $N/\Gamma \times_{\tilde{f}_k} S_1$ can be given the structure of a j-step nilflow. If the nilflow $(N/\Gamma, a)$ has a finite factor C_i, then C_i is an abelian group of order i for some integer i. The nilflow $(N/\Gamma, a^i)$ has finitely many ergodic components, and rotation by a induces an isomorphism between them. Each ergodic component X (with the action of a^i) is a $j-1$ step nilflow with no nontrivial finite factors. We will show that the system $X \times_{\tilde{f}_k} S^1$ isomorphic to a j-step nilflow $(M/\Lambda, b)$. The system $N/\Gamma \times_{\tilde{f}_k} S^1$ will then be isomorphic to a union of i isomorphic j-step nilflows $C_i \times M/\Lambda$, with the action of T given by: for $i - 1 > k \geq 0 : (k, m\Lambda) \to (k+1, m\Lambda)$ and for $k = i - 1 :$ $(i-1, m\lambda) \to (0, bm\Lambda)$. The group generated by $\{(0, m), T\}_{m \in M}$ is j-step nilpotent, and acts transitively on $C_i \times M/\Lambda$.

Assume $(N/\Gamma, a)$ has no finite non trivial factors. The r-u.c.f of N/Γ is $Y_r(N/\Gamma) = N/N_r\Gamma$ (this follows from lemma 4.5). The system $(N/\Gamma, a)$ can be presented as an abelian extension of a $(j-2)$-step nilflow i.e. $(N/N_{j-1}\Gamma \times N_{j-1}/(N_{j-1} \cap \Gamma)$, and we may assume that N is simply connected. The group N_{j-1} is abelian, connected and simply connected ([L02]), therefore isomorphic to $\mathbb{R}^m$ for some m. Let the action of T on $N/N_2 \cong \mathbb{Z} \times \mathbb{R}^n$ be given by translation by α. Then equation (36) becomes: for any $c \in N_{j-1}$,

$k = 1, \dots, j+1$, $y \in N/\Gamma$

$$\frac{\tilde{f}_k(cy)}{\tilde{f}_k(y)} = \lambda_{k,c} \frac{T f_{k,c}(y)}{f_{k,c}(y)}. \tag{37}$$

6.18. **Lemma.** *Let $(N/\Gamma, a)$ be the $(j-1)$-step nilflow from 6.17.*

Let $f_1, \dots, f_k$ be functions in $B(N/\Gamma, S^1)$. Let $\{\lambda_{k,c}\}$ be a family of constants, $\{f_{k,c}\}_{c \in N_{j-1}}$ be a family of functions in $B(N/\Gamma, S^1)$ such that

$$\frac{f_k(cy)}{f_k(y)} = \lambda_{k,c} \frac{f_{k,c}(ay)}{f_{k,c}(y)} \tag{38}$$

for all $c \in N_{j-1}$. Then We can choose $f_{k,c}, \lambda_{k,c}$ in equation (38) such that

$$f_{k,c_1}(k, c_2 y) f_{k,c_2}(y) \sim f_{k,c_1 c_2}(y).$$

Proof. It follows from lemma 4.7. □

6.19. **Lemma.** *Let $f_k, f_{k,c}, \lambda_{k,c}$ be from lemma 6.18. Then for each $k = 1, \dots j+1$ there exists an integer matrix L_k, a neighborhood of zero $U \subset N_{j-1}$, and a family of functions $\{f_{k,c}\}_{c \in U}$, such that for all $c \in U$*

$$\frac{f_k(cy)}{f_k(y)} = e^{2\pi i \langle L_k \alpha, c \rangle} \frac{f_{k,c}(ay)}{f_{k,c}(y)}.$$

Proof. It follows from lemma 4.8. □

6.20. **Corollary.** *Let Y be a $(j-1)$-step pro-nilflow, $(f_1, \dots, f_{j+1})$ of type $j+1$ w.r.t. $\bar{\triangle}_{j+1}(\mu_Y)$. Then modulo quasi-coboundaries, f_k belongs to a countable set.*

Proof. By lemma 6.15, f_k is cohomologous to a function $\tilde{f}_k$ lifted from a nilflow, and by remark 6.16 there are countably many possibilities for this nilflow. Fix the nilflow. If $(\tilde{f}_1, \dots, \tilde{f}_{j+1})$, $(\tilde{f}'_1, \dots, \tilde{f}'_{j+1})$ share the integer matrices $L_1, \dots, L_{j+1}$ from lemma 6.19, then by corollary 3.9, $\tilde{f}_k/\tilde{f}'_k$ is cohomologous to a function on $\pi_{j-1}(N/\Gamma)$. By lemma 6.8, as a function on $\pi_{j-1}Y$, $\tilde{f}_k/\tilde{f}'_k$ is of type $j-1$. By the induction hypothesis in theorem 6.1(3b), $\tilde{f}_k/\tilde{f}'_k$ belongs to a countable set modulo $\pi_{j-1}Y$ quasi-coboundaries. (If we use condition (2) in lemma 6.15 then we can get that $\tilde{f}_k/\tilde{f}'_k$ belongs to a countable set modulo $\pi_{j-1}(N/\Gamma)$ quasi-coboundaries). □

6.21. **Lemma.** *Let Y be a $(j-1)$-step pro-nilflow, and let $(f_1, \dots, f_l)$ be of type l w.r.t $\bar{\triangle}_l(\mu_Y)$. Then for each $k = 1, \dots, l$ there exists a family of constants $\{\lambda_{k,u}\}_{u \in H_j}$, and a family of functions $\{f_{k,u}\}_{u \in H_j} \subset B(Y^0, S^1)$ such that*

$$\frac{f_k(\pi_{j-1}y, h+u)}{f_k(\pi_{j-1}y, h)} = \lambda_{k,u} \frac{T f_{k,u}(y)}{f_{k,u}(y)}. \tag{39}$$

Proof. We use induction on l. The proof for $l \leq j+1$ is given in corollary 6.10. Assume the statement holds for l; we show it for $l+1$. Let $(f_1, \dots, f_{l+1})$ be of type $l+1$ w.r.t $\bar{\triangle}_{l+1}(\mu_Y)$, and let $M_{j-1,l+1}(0) = (M_{j-1,l+1} + \triangle_{l+1}(H_j)) \cap H_j^l \times \{0\}$. Let $\vec{u} = (u_1, \dots, u_{l+1}) \in M_{j-1,l+1}(0)$ $(u_{l+1} = 0)$. Then

$$\left(\frac{f_1(\pi_{j-1}y, h+u_1)}{f_1(\pi_{j-1}y, h)}, \dots, \frac{f_l(\pi_{j-1}y, h+u_l)}{f_l(\pi_{j-1}y, h)}\right)$$

is of type l and by the induction hypothesis

$$\frac{f_k(\pi_{j-1}y, h+u_k+u)/f_k(\pi_{j-1}y, h+u)}{f_k(\pi_{j-1}y, h+u_k)/f_k(\pi_{j-1}y, h)} = \lambda_{k,u,u_k}\frac{Tf_{k,u,u_k}(y)}{f_{k,u,u_k}(y)}.$$

The projection of $M_{j-1,l+1}(0)$ on any coordinate $k \leq l$ is full. By lemma 6.11, fixing u, λ_{k,u,u_k} is multiplicative in u_k in a neighborhood of zero in H_j. By lemma 6.19, λ_{k,u,u_k} is determined by an integer matrix. The same holds interchanging the roles of u and u_k. Therefore $\lambda_{k,u,u_k} \equiv 1$ in a neighborhood of zero in H_j. By lemma 3.9

$$\frac{f_k(\pi_{j-1}y, h+u)}{f_k(\pi_{j-1}y, h)} = g_{k,u}(\pi_{j-1}y)\frac{Tf_{k,u}(y)}{f_{k,u}(y)}.$$

As in lemma 6.8 and corollary 6.10, the functions $g_{k,u}(\pi_{j-1}y)$, $f_{k,u}(y)$ can be chosen so that $g_{k,u}(\pi_{j-1}y)$ is a constant function on Y_{j-1}. □

6.22. **Corollary.** *We may replace the index $j+1$ in lemmas/corollaries 6.7 - 6.20 by the index l for any $l \geq j+1$.*

As a corollary we get

6.23. *Proof of theorem 1.7(3a)*: It follows from lemma 6.15 and corollary 6.22.

6.24. *Proof of theorem 1.7(3b)*: It follows from corollaries 6.20 and 6.22.

6.25. We now fix N/Γ - a $(j-1)$-step nilflow as in 6.17. Let $(f_1, \dots, f_l)$ be of type l w.r.t $\bar{\triangle}_l(\mu_{N/\Gamma})$. We will show that the system $N/\Gamma \times_{f_k} S^1$ is isomorphic to a j step nilflow. We have constructed families of functions $\{f_{k,c}\}_{c \in N_{j-1}}$ satisfying equation (38).

6.26. **Lemma.** *For any $c_1, c_2 \in N_{j-1}$, f_{k,c_1} and f_{k,c_2} satisfying equation (38) we have*

$$\frac{f_{k,c_1}(c_2 y)}{f_{k,c_1}(y)} = \frac{f_{k,c_2}(c_1 y)}{f_{k,c_2}(y)}$$

Proof. It follows from lemma 4.9.

□

6.27. Consider the group

$$\mathcal{G} = \{(n, f) : \ n \in N, \ f \in B(N/\Gamma, S^1)\},$$

with multiplication

$$(n, f)(m, g) = (nm, f^m g), \qquad (f^m g)(y) = f(my)g(y).$$

The elements of the form $(1, C)$ where C is a constant are in $Z(\mathcal{G})$ - the center of $\mathcal{G}$. For $c \in N_{j-1}$, we can interpret equation (38) as

$$[(a, f), (c, f_c)] = (1, \lambda_c) \in Z(\mathcal{G})$$

We want to think of f_c as elements of the $(j-1)$th subgroup in the upper central series of $\mathcal{G}$, which will be a j-step nilpotent group. We now follow the derived series upward to construct for each element in N a function $f_{k,n}$ that will satisfy good commuting relations with (a, f_k).

6.28**. Notation.** For $n \in N_i \backslash N_{i+1}$, $|n| = i$, and $f^n(y) = f(ny)$

6.29**. Proposition.** *Let $Y = (N/\Gamma, a)$ be a $(j-1)$-step nilflow. Let $(f_1, \ldots, f_l)$ be of type j w.r.t $\bar{\triangle}_l(\mu_Y)$. For $k = 1, \ldots, l$ there exists a family of functions $\mathcal{F}_k = \{f_{k,n}\}_{n \in N}$, where $f_{k,n} : N/\Gamma \to S^1$ are measurable functions satisfying the following conditions:*

(1)
$$\frac{f_k(ny)}{f_k(y)} \sim f_{k,[a,n]}(nay) \frac{f_{k,n}(ay)}{f_{k,n}(y)},$$

(2) *For any $c \in N_{j-1}$, $|n| > 1$:*
$$f^n_{k,c} f_{k,n} = f^c_{k,n} f_{k,c}.$$

(3)
$$\frac{f_{k,n_1}(n_2 y)}{f_{k,n_1}(y)} \sim f_{k,[n_1,n_2]}(n_2 n_1 y) \frac{f_{k,n_2}(n_1 y)}{f_{k,n_2}(y)},$$

(4)
$$f_{k,n_1 n_2}(y) \sim f_{k,n_1}(n_2 y) f_{k,n_2}(y).$$

(5) *If $(n_1, \ldots, n_l) \subset N^l$ preserves the ergodic components of $\tau_l(Y)$ then*
$$\frac{F(n_1 y_1, \ldots, n_l y_l)}{F(y_1, \ldots, y_l)} \prod_{k=1}^{l} \bar{f}_{k,n_k}(y_k)$$
is constant $\bar{\triangle}_l(\mu_Y)$ a.e.

Proof. The proof requires a series of lemmas and their corollaries and will be completed in 6.36. We prove this inductively, proceeding upward in the derived series of N. By lemmas 6.19, 6.18, 6.26, and corollary 6.12, conditions (1)-(4) hold for all $n \in N_{j-1}$. As $N_{j-1} \subset Z(N)$ the function in (5) is invariant under τ_l, T_l. Suppose we constructed f_n for n in N_{i+1} $(i + 1 > 1)$ satisfying conditions (1)-(5). Let $n \in N_i$. Using conditions (1),(2), for $c \in N_{j-1}$ we have:

$$\frac{f_k(ncy)}{f_k(cy)} \bar{f}_{k,[a,n]}(nacy) \Bigg/ \frac{f_k(ny)}{f_k(y)} \bar{f}_{k,[a,n]}(nay) = \frac{f_{k,c}(nay)}{f_{k,c}(ay)} \Bigg/ \frac{f_{k,c}(ny)}{f_{k,c}(y)} .$$

By corollary 3.9 there exist functions $f_{k,n} : N/\Gamma \to S^1$, and $g_{k,n} : N/(N_{j-1}\Gamma)$ $\to S^1$ such that

$$\frac{f_k(ny)}{f_k(y)} \bar{f}_{k,[a,n]}(nay) = g_{k,n}(\pi_{j-1} y) \frac{f_{k,n}(ay)}{f_{k,n}(y)}. \tag{40}$$

Let $c \in N_{j-1}$. By the induction hypothesis $f_{k,c}^{[a,n]} f_{[a,n]} = f_{[a,n]}^{c} f_{k,c}$ therefore

$$\frac{f_{k,c}(ny) f_{k,n}(y)}{f_{k,c}(y) f_{k,n}(cy)}$$

is a T invariant function and therefore constant $\delta(k, n, c)$.

6.30. **Lemma.** *The l-tuple $(g_{1,n}, \ldots, g_{l,n})$ is of type l w.r.t $\bar{\triangle}_l(\mu_Y)$.*

Proof. Iterating equation (40) (using condition (4)) we get in a neighborhood of zero in N_i

$$\frac{f_k^{(k)}(ny)}{f_k^{(k)}(y)} \bar{f}_{k,[a^k,n]}(na^k y) = g_{k,n}^{(k)}(\pi_{j-1} y) \frac{f_{k,n}(a^k y)}{f_{k,n}(y)}.$$

Substituting in the functional equation (46) we get

$$\begin{aligned}
&\prod_{k=1}^{l} \left((g_{k,n}^{(k)}(\pi_{j-1} y_k) f_{k,[a^k,n]}(na^k y_k) \frac{f_{k,n}(a^k y_k)}{f_{k,n}(y_k)} \right) \\
&= \frac{F(any_1, \ldots, a^l n y_l) / F(ny_1, \ldots, ny_l)}{F(ay_1, \ldots, a^l y_l) / F(y_1, \ldots, y_l)} \\
&= \frac{F([a,n]nay_1, \ldots, [a^l, n]na^l y_l)}{F(nay_1, \ldots, na^l y_l)} \frac{G_n(ay_1, \ldots, a^l y_l)}{G_n(y_1, \ldots, y_l)},
\end{aligned}$$

where

$$G_n(y_1, \ldots, y_l) = F(ny_1, \ldots, ny_l) / F(y_1, \ldots, y_l).$$

By induction using condition (5)

$$\frac{F([a,n]nay_1, \ldots, [a^l, n]na^l y_l)}{F(nay_1, \ldots, na^l y_l)} \prod_{k=1}^{l} \left(\bar{f}_{k,[a^k,n]}(na^k y_k) \right)$$

is constant $\bar{\triangle}_l(\mu_Y)$ a.e. $\square$

6.31. **Corollary.** *For $k = 1, \ldots, l$, $g_{k,n}$ is of type $j-1$, therefore the set $\{g_{k,n}\}_{n \in N_i}$ modulo $\pi_{j-1} Y$-quasi-coboundaries is countable.*

Proof. By lemma 6.8. $\square$

6.32. **Corollary.** *Suppose for some $1 \leq r \leq j-1$, $g_{k,n}(y) = g_{k,n}(\pi_r y)$ for $k = 1, \ldots, l$. Then $g_{k,n}(\pi_r y)$ is of type r, and therefore the set $\{g_{k,n}(\pi_r y)\}_{n \in N_i}$ modulo $\pi_r Y$-quasi-coboundaries is countable.*

6.33. **Example.** Before proceeding with the general proof we describe the proof for the case where Y is a homogeneous space of a 3-step nilpotent group (i.e $N_4 = 1$, $j-1 = 3$), $f_k : Y \to S^1$. Let $n, m \in N_2$. We use the facts that $[a, m] \in N_3$, therefore

$$f_{k,[a,m]}^{n} f_{k,n} \sim f_{k,n}^{[a,m]} f_{k,[a,m]},$$

and that for $m_1, m_2 \in N_3$,

$$f_{k,m_1m_2} \sim f_{k,m_1}^{m_2} f_{k,m_2}.$$

Recall that by corollary 6.31 there are countably many $g_{k,n}$ up to $\pi_3 Y$-quasi-coboundaries. Therefore there exists $m \in N_2$ so that for n in a neighborhood of zero in N_2 $(g_{k,nm}/g_{k,m})(\pi_3 y)$ is a quasi-coboundary:

$$(g_{k,nm}/g_{k,m})(\pi_3 y) \sim L_{k,n,m}(a\pi_3 y)/L_{k,n,m}(\pi_3 y)$$

Fix n and replace $f_{k,m}(y)$ with $f_{k,m}(y)L_{k,n,m}(\pi_3 y)$ (this does not effect the commutation relations with $(c, f_{k,c})$ for $c \in N_3$). Now $g_{k,nm}/g_{k,m}$ is a constant. Then

$$\begin{aligned}\frac{f_k(nmy)}{f_k(my)} &= \frac{f_k(nmy)/f_k(y)}{f_k(my)/f_k(y)} \sim \frac{f_{k,[a,nm]}(nmay)}{f_{k,[a,m]}(may)} \frac{f_{k,nm}(am^{-1}my)/f_{k,nm}(m^{-1}my)}{f_{k,m}(am^{-1}my)/f_{k,m}(m^{-1}my)} \\ &\sim f_{k,[n,a]}(many)\frac{\tilde{f}_{k,n}(amy)}{\tilde{f}_{k,n}(my)}.\end{aligned}$$

Therefore $g_{k,n}$ may be chosen to be constant (with a proper choice of $f_{k,n}$). Iterating, this holds for all $n \in N_2$. To show that we can further modify $f_{k,n}$ so that $f_{k,nm} \sim f_{k,n}^m f_{k,n}$ we observe that

$$1 = \frac{f_k(nmy)}{f_k(y)} \Big/ \frac{f_k(nmy)}{f_k(my)}\frac{f_k(my)}{f_k(y)} \sim \frac{f_{k,nm}(ay)\,/f_{k,n}(may)f_{k,m}(ay)}{f_{k,nm}(y)\,/f_{k,n}(my)f_{k,m}(y)}. \tag{41}$$

Now proceed as in lemma 3.15. As a corollary we get that for $c \in N_3$, $n \in N_2$ we have $f_{k,n}^c f_{k,c} = f_{k,c}^n f_{k,n}$ (same proof as lemma 6.26). Now let $n \in N_1 = N$. We construct $f_{k,n}, g_{k,n}$ as in equation (40), and as in corollary 6.31 $g_{k,n}$ is of type 3 and there are countably many $g_{k,n}$ up to $\pi_3 Y$-quasi-coboundaries. We first need to establish commutation relations between $(n, f_{k,n})$ and $(m, f_{k,m})$ for $m \in N_3$:

$$\frac{f_{k,n}(my)}{f_{k,n}(y)} \Big/ \frac{f_{k,m}(ny)}{f_{k,m}(y)}$$

is a T invariant function and therefore a constant $\delta(k, n, m)$, which is multiplicative in m. We use this to establish commutation relations between $(n, f_{k,n})$ and $(m, f_{k,m})$ for $m \in N_2$:

$$\begin{aligned}&\frac{T(\frac{f_{k,n}(my)}{f_{k,n}(y)}/\frac{f_{k,m}(ny)}{f_{k,m}(y)}f_{k,[n,m]}(mny))}{(\frac{f_{k,n}(my)}{f_{k,n}(y)}/\frac{f_{k,m}(ny)}{f_{k,m}(y)}f_{k,[n,m]}(mny))} \\ =&\delta(k, n, [a, m])\delta(k, a, [m, n])\delta(k, m, [n, a])\frac{g_{k,n}(\pi_3 my)}{g_{k,n}(\pi_3 y)} \\ =&\delta(k, n, [a, m])\delta(k, a, [m, n])\delta(k, m, [n, a])c(k, n, m)\frac{h_{k,n,m}(\pi_3 ay)}{h_{k,n,m}(\pi_3 y)},\end{aligned}$$

Where $c(k,n,m)$ is multiplicative in m ($g_{k,n}$ is of type 3 therefore satisfies equation 38 for $m \in N_2$). Thus for some eigenfunction $\psi_{k,n,m}(\pi_2 y)$

$$\frac{f_{k,n}(my)}{f_{k,n}(y)}\bar{f}_{k,[n,m]}(nay) = \psi_{k,n,m}(y)h_{k,n,m}(\pi_3 y)\frac{f_{k,n}(ay)}{f_{k,n}(y)}. \tag{42}$$

Let $m,n \in N$. As before we find that

$$\frac{g_{k,nm}(\pi_3 y)}{g_{k,n}(\pi_3 my)g_n(\pi_3 y)}h_{k,n,[m,a]}(\pi_3 amy) \sim \frac{T(f_{k,nm}(y)/f_{k,n}(my)f_{k,m}(y))}{f_{k,nm}(y)/f_{k,n}(my)f_{k,m}(y)}.$$

Same proof as lemma 3.15 shows that $f_{k,n}, f_{k,m}$ can be chosen so that for m,n in a neighborhood of zero in N, there exists a function $K_{n,m}(\pi_3 y)$ such that

$$\frac{f_{k,nm}(y)}{f_{k,n}(my)f_{k,m}(y)} = K_{n,m}(\pi_3 y).$$

This implies that $\delta(k,n,c)$ is multiplicative for $n \in N$: on the one hand

$$\frac{f_{k,n_1n_2}(cy)}{f_{k,n_1n_2}(y)} = \delta(k,n_1n_2,c)\frac{f_{k,c}(n_1n_2y)}{f_{k,c}(y)},$$

while on the other hand

$$\frac{f_{k,n_1n_2}(cy)}{f_{k,n_1n_2}(y)} = \frac{f_{k,n_1}(n_2cy)f_{k,n_2}(cy)}{f_{k,n_1}(n_2y)f_{k,n_2}(y)} = \delta(k,n_1,c)\delta(k,n_2,c)\frac{f_{k,c}(n_1n_2y)}{f_{k,c}(y)}.$$

Therefore the constant $\delta(k,n,[a,m])\delta(k,a,[m,n])\delta(k,m,[n,a])$ is multiplicative in m,n and must equal 1 (as $c(k,n,m)$ is locally constant as a function of n). Thus $c(k,n,m)$ is an eigenvalue of T, but it is multiplicative in m in a neighborhood of zero in N_2, therefore $c(k,n,m) \equiv 1$. This implies that $g_{k,n}$ is cohomologous to a function on $N/N_2\Gamma$, and we can chose $f_{k,n}$ so that $g_{k,n}$ is lifted from a function on $\pi_2 Y$. Then $(n, f_{k,n})$ $(m, f_{k,m})$ commute nicely for $m \in N_2$; namely $[(n,f_{k,n}),(m,f_{k,m})] = ([n,m], cf_{[n,m]})$ for some constant c. Let $n,m \in N$ be so $(g_{k,nm}/g_{k,m})(\pi_2 y) \sim L_{k,n,m}(a\pi_2 y)/L_{k,n,m}(\pi_2 y)$ (by corollary 6.32 there are countably many $g_{k,n}$ up to $\pi_2 Y$-quasi-coboundaries) Replace $f_{k,m}(y)$ with $f_{k,m}(y)L_{k,n,m}(\pi_2 y)$ (this does not effect the commutation relations with $(l, f_{k,l})$ for $l \in N_2$). Now $g_{k,nm}/g_{k,m}$ is a constant. Computation (using the fact that $(n, f_{k,n}), (nm, f_{k,nm})$ and $([a,n^{-1}], f_{k,[a,n^{-1}]})$ commute nicely) shows (as before):

$$\frac{f_k(nmy)}{f_k(my)} = \frac{f_k(nmy)/f_k(y)}{f_k(my)f_k(y)} \sim f_{k,[n,a]}(namy)\frac{\tilde{f}_{k,n}(amy)}{\tilde{f}_{k,n}(my)}.$$

And continue as in lemma 3.15 to find that $f_{k,nm} \sim f_{k,n}^m f_{k,m}$.

We now proceed with the induction in proposition 6.29. Basically we follow the same procedure.

6.34. **Lemma.** *Let $n \in N_i$. The functions g_n and f_n can be chosen so that g_n is a constant function on Y.*

Proof. We already know that for $c \in N_{j-1}$ we have $f_{k,n}^c f_{k,c}/f_{k,c}^n f_{k,n}$ is a constant denoted $\delta(k,n,c)$. We use induction on $i+1 \leq r \leq j$ to acquire 'good' commuting relations between $(n, f_{k,n}), (m, f_{k,m})$ for $m \in N_r$, and to reduce the 'level' of $g_{k,n}$. Assume that

(1) $g_{k,n}$ is lifted from $N/N_{r+1}\Gamma$ and is of type $r+1$.
(2) For $m \in N_{r+1}$,
$$f_{k,n}^m f_{k,m} = \delta(k,n,m) f_{k,[m,n]} f_{k,m}^n f_{k,n},$$
(3) For $m \in N_{r+1}$,
$$\begin{aligned}\delta(k,n_1n_2,m) f_{k,[m,n_1n_2]} =& \delta(k,n_1,[n_2,m])\delta(k,n_1,m)\delta(k,n_2,m)\\ & f_{k,[[m,n_1],n_2]}^{[n_1,m][n_2,m]} f_{k,[m,n_1]}^{[n_2,m]} f_{k,[m,n_2]},\end{aligned}$$
and similarly for $\delta(k,n,m_1m_2)$.
(4) $f_{k,n_1n_2} = f_{k,n_1}^{n_2} f_{k,n_2}(y) K_{k,n_1,n_2}(\pi_{r+1}(y))$.

Let $m \in N_r$. Calculations using the induction hypothesis, and commutator identities lead to

$$\begin{aligned}&\frac{T(\frac{f_{k,n}(my)}{f_{k,n}(y)}/\frac{f_{k,m}(ny)}{f_{k,m}(y)} f_{k,[n,m]}(mny))}{(\frac{f_{k,n}(my)}{f_{k,n}(y)}/\frac{f_{k,m}(ny)}{f_{k,m}(y)} f_{k,[n,m]}(mny))}\\ =&\Lambda(k,n,m)\frac{g_{k,n}(\pi_{r+1}my)}{g_{k,n}(\pi_{r+1}y)}\\ =&\Lambda(k,m,n)c(k,n,m)\frac{h_{k,n,m}(\pi_{r+1}ay)}{h_{k,n,m}(\pi_{r+1}y)},\end{aligned} \tag{43}$$

where

$$\Lambda(k,m,n_1n_2) = \prod_{q=1}^{Q} \Lambda(k, p_{1,q}(m,n_1,n_2), p_{2,q}(m,n_1,n_2))$$

where $p_{1,q}(m,n_1,n_2) \in N_r$ is an expression involving m, m^{-1} and commutators involving m, n_1, n_2, a and their inverses, and $p_{2,p}(m,n_1,n_2) \in N_i$ is an expression involving m, n_1, n_2, a and there inverses - this is due to condition (3) (Same holds for $\Lambda(k, m_1m_2, n)$), and $c(k,n,m)$ is multiplicative in m ($g_{k,n}$ is of type $r+1$). As $g_{k,n}$ is countably determined up to quasi-coboundaries $c(k,n,m) \equiv 1$. Therefore $g_{k,n}(m\pi_{r+1}y)/g_{k,n}(\pi_{r+1}y)$ is a $\pi_{r+1}Y$-coboundary for any $m \in N_r$, hence $g_{k,n}(\pi_{r+1}y)$ is $\pi_{r+1}Y$-cohomologous to a function on $\pi_r Y$. Therefore we can choose $g_{k,n}, f_{k,n}$ such that $g_{k,n}$ is invariant under $m \in N_r$. This implies that

$$f_{k,n}^m f_{k,m} = \delta(k,n,m) f_{k,[m,n]} f_{k,m}^n f_{k,n}$$

($h_{k,n,m}$ is the constant function 1). Same calculation as in equation (43) gives for any $i < s$, $m \in N_s$

$$f_{k,n}^m f_{k,m} = h_{k,n,m}(\pi_r y) f_{k,[m,n]} f_{k,m}^n f_{k,n}.$$

Using this we get, as in lemma 3.15, that the functions $f_{k,n}$ can be chosen so that for $n_1, n_2 \in N_i$ there exists a function $K_{k,n_1,n_2}(\pi_r y)$ so that

$$f_{k,n_1n_2}(y) = f^{n_2}_{k,n_1} f_{k,n_2}(y) K_{k,n_1,n_2}(\pi_r y).$$

To get the condition on $\delta(k,n,m)$ for $m \in N_r$: on the one hand

$$\frac{f_{k,n_1n_2}(my)}{f_{k,n_1n_2}(y)} = \delta(k, n_1n_2, m) f_{k,[n,m]}(mny) \frac{f_{k,m}(n_1n_2y)}{f_{k,m}(y)},$$

while on the other hand

$$\begin{aligned}\frac{f_{k,n_1n_2}(my)}{f_{k,n_1n_2}(y)} =& \frac{f_{k,n_1}(n_2my) f_{k,n_2}(my)}{f_{k,n_1}(n_2y) f_{k,n_2}(y)} \\ =& \delta(k,n_1,m)\delta(k,n_2,m)\delta(k,n_1,[n_2,m]) f_{k,[n_1,m]}(mn_1n_2y) \\ & f_{k,[n_2,m]}(n_1mn_2y) f_{k,[n_1,[n_2,m]]}([n_2,m]n_1mn_2y) \frac{f_{k,m}(n_1n_2y)}{f_{k,m}(y)}.\end{aligned}$$

Finally, having good commutation relations between $(n, f_{k,n}), (m, f_{k,m})$ for $n \in N_i$, $m \in N_{i+1}$ (i.e. $[(n, f_{k,n}), (m, f_{k,m})] = ([n,m], \delta(k,n,m) f_{k,[n,m]})$), we show that for $n \in N_i$, the functions $f_{k,n}, g_{k,n}$ can be chosen so that $g_{k,n}$ is constant. we already know it is lifted from $\pi_{i+1}Y$. Let $n, m \in N_i$ be so that

$$(g_{k,nm}/g_{k,m})(\pi_{i+1}y) \sim h_{k,n,m}(a\pi_{i+1}y)/h_{k,n,m}(\pi_{i+1}y)$$

(recall that by corollary 6.32 there are countably many $g_{k,n}$ up to $\pi_{i+1}Y$-quasi-coboundaries). Replace $f_{k,m}(y)$ with $f_{k,m}(y)h_{k,n,m}(\pi_{i+1}y)$ (this does not affect the commutation relations with $f_{k,p}$ for $p \in N_{i+1}$). Now $g_{k,nm}/g_{k,m}$ is a constant. Computation shows:

$$\frac{f_k(nmy)}{f_k(my)} = \frac{f_k(nmy)/f_k(y)}{f_k(my)/f_k(y)} \sim f_{k,[n,a]}(namy) \frac{\tilde{f}_{k,n}(amy)}{\tilde{f}_{k,m}(my)}.$$

And continue in lemma 3.15 to find that $f_{k,nm} \sim f^m_{k,n} f_{k,m}$. □

6.35. Corollary. *If $i > 1$ then for any $c \in N_{j-1}$, $n \in N_i$*

$$f_{k,n}(cy) f_{k,c}(y) = f_{k,c}(ny) f_{k,n}(y)$$

(i.e. $[(n, f_{k,n}), (c, f_{k,c})] = (1,1)]$).

Proof. The quotient

$$f_{k,n}(cy) f_{k,c}(y) / f_{k,c}(ny) f_{k,n}(y)$$

is invariant under translation by a and therefore a constant $\delta(k,n,c)$ which is multiplicative in both coordinates. For $\gamma \in N_{j-1}$, f_γ is an eigenfunction and therefore invariant under the action of N_i for $i > 1$. This implies that $\delta(k,n,c\gamma) = \delta(k,n,c)$. Proceed as in lemma 6.26. □

6.36. Corollary. *If $(n_1, \ldots, n_l) \subset N_i^l$ preserve the ergodic components of $\tau_l(Y)$ then*

$$\frac{F(n_1y_1, \ldots, n_ly_l)}{F(y_1, \ldots, y_l)} \prod_{k=1}^{l} \bar{f}_{k,n_k}(y_k) \tag{44}$$

is constant $\bar{\triangle}_l(\mu_Y)$ *a.e.*

Proof. Both $([a, n_1], \ldots, [a, n_l])$, and $([a, n_1], \ldots, [a^l, n_l])$ preserve the ergodic components of τ_l (see 4.1), and as $g_{k,n}$ is constant for $n \in N_i$ the function in the left hand of equation (44) is invariant under τ_l, T_l (this is a calculation, using condition (1)). □

The proof of proposition 6.29 is now complete. □

6.37. **Proposition.** *Let* $Y = N/\Gamma$ *be a* $(j-1)$*-step nilflow, and let* $(f_1, \ldots, f_l)$ *be of type* l *w.r.t* $\bar{\triangle}_l(\mu_Y)$*. Then for any* $k = 1, \ldots, l$ *the system* $Y \times_{f_k} S^1$ *can be given the structure of a* j*-step nilflow.*

Proof. Denote

$$\mathcal{G}_k = \{(n, \psi f_{k,n}) : \ n \in N, \psi \text{ an eigenfunction}, \ f_{k,n} \in \mathcal{F}_k\}.$$

where $\mathcal{F}_k$ is defined in proposition 6.29. $\mathcal{G}_k$ is a group under the multiplication

$$(n, f)(m, g) = (nm, f^m g), \qquad (f^m g)(y) = f(my)g(y).$$

By proposition 6.29, $\mathcal{G}_k$ is a j-step nilpotent group, and $(a, f_k) \in \mathcal{G}_k$. Endow $\mathcal{G}_k$ with the topology:

$$(n_i, g_i) \to (n, g) \iff n_i \to n, \ g_i \overset{L^2(N/\Gamma)}{\longrightarrow} g.$$

$\mathcal{G}_k$ acts transitively and effectively on $X = N/\Gamma \times S^1$ by:

$$(n, f)(y, \zeta) = (ny, f(y)\zeta).$$

By a theorem of Montgomery and Zippin (see [GOV97] page 88, Theorem 4.3) it possesses a Lie group structure. This type of construction is carried out in [Me90] for the case $j = 2$. □

6.38. **Remark.** It may also be possible to construct the nilflow is using the constants $\delta(k, n, m)$ as was done in [R93],[Le93] for 2-step nilpotent groups, and in [Z02b] for 3-step nilpotent groups.

6.39. **Lemma.** *Let* $Y = N/\Gamma$ *be a* $(j-1)$*-step nilflow, and let* $f : Y \to S^1$ *be of type* j*. Then the system* $Y \times_f S^1$ *can be given the structure of a* j*-step nilflow.*

Proof. By definition, for $k = 1, \ldots, l$, there exist integers m_k, with $m_k = 1$ for some k, and $(f^{m_1}, \ldots, f^{m_l})$ of type l w.r.t $\bar{\triangle}_l(\mu_Y)$. Now use proposition 6.37.

□

A similar proof gives:

6.40. **Lemma.** *Let* $Y = N/\Gamma$ *be a* $(j-1)$*-step nilflow, and let* $f, g : Y \to S^1$ *be of type* j*. Then the system* $Y \times_{fg} S^1$ *can be given the structure of a* j*-step nilflow.*

6.41. **Corollary.** *Let Y be a $(j-1)$-step pro-nilflow, and let $(f_1,\dots,f_l)$ be of type l w.r.t $\bar{\triangle}_l(\mu_Y)$. Then for any $k=1,\dots,l$ the system $Y\times_{f_k}S^1$ can be given the structure of a j-step pro-nilflow.*

Proof. By lemma 6.15 and corollary 6.22, f_k is cohomologous to a cocycle $\tilde{f}_k$ lifted from a $(j-1)$-step nilflow $(N/\Gamma,a)$. Furthermore, there exist $(g_1,\dots,g_l)$ with g_k of type $j-1$, such that $(\tilde{f}_1g_1,\dots,\tilde{f}_lg_l)$ is of type l w.r.t $\bar{\triangle}_l(\mu_{N/\Gamma})$. By proposition 6.37, $N/\Gamma\times_{\tilde{f}_kg_k}S^1$ can be given the structure of a j-step nilflow. By lemma 6.39, $N/\Gamma\times_{g_k^{-1}}S^1$ can be given the structure of a j-step nilflow. By the construction in proposition 6.37, $N/\Gamma\times_{\tilde{f}_kg_kg_k^{-1}}S^1$ can be given the structure of a j-step nilflow. □

6.42. *proof of 6.1(4)*

Let $X=Y_j(X)\times_\rho H$, where H is a compact abelian group, and for any $\chi\in\hat{H}$, there exists $(\chi_1,\dots,\chi_l)\in\hat{H}^l$, with $\chi=\chi_k$ for some k, and $(\chi_1\circ\rho,\dots,\chi_l\circ\rho)$ of type $\vec{a}$ w.r.t $\bar{\triangle}_{\vec{a}}(Y_j(X))$. By corollary 6.41 the system $X=Y_j(X)\times_{\chi\circ\rho}S^1$ is isomorphic to a j-step pro-nilflow. By Pontryagin duality, $H\hookrightarrow(S^1)^{\hat{H}}$. The system $Y\times_\rho H$ is therefore a 'join' of factors of the form $Y\times_{\chi\circ\rho}S^1$ where χ ranges over $\hat{H}$.

6.43. **Lemma.** *Let Y be a $(j-1)$-step pro-nilflow, and let $(f_1,\dots,f_l)$ be of type l w.r.t $\bar{\triangle}_l(\mu_Y)$. If for some k, $\lambda_{k,u}\equiv 1$ (see lemma 6.21) in a neighborhood of zero in H_j, then the system $Y\times_{f_k}S^1$ can be given the structure of a $(j-1)$-step pro-nilflow.*

Proof. By the induction hypothesis 6.1(6), H_j is connected $(j\geq 1)$ therefore we may choose $\lambda_{k,u}\equiv 1$ on H_j. By corollary 3.9, f_k is cohomologous to a cocycle $\tilde{f}_k$ lifted from Y^0_{j-1}. By lemma 6.8, $\tilde{f}_k$ is of type $j-1$. By the induction hypothesis 6.1(4), $Y_{j-1}\times_{\tilde{f}_k}S^1$ can be given the structure of a $(j-1)$-step pro-nilflow. □

6.44. **Lemma.** *Let $X=Y_j(X)\times_\rho H$, where H is a compact abelian group, and ρ of type j. Then H is connected.*

Proof. We show that $\hat{H}$ has no elements of finite order. Suppose for some $\chi\in\hat{H}$, and some $l>0$, $\chi^l=1$. $\chi\circ\rho$ satisfies equation (36). By lemma 6.11 the function $\lambda_{k,u}$ is multiplicative in a neighborhood of zero in H_j. $\lambda^l_{k,u}$ is an eigenvalue, and as H_j is connected $\lambda_{k,u}\equiv 1$ in a neighborhood of zero. By lemma 6.43 the system $Y'_j=Y_j(X)\times_{\chi\circ\rho}S^1$, which is a factor of $X=Y_j(X)\times_\rho H$, can be given the structure of a $(j-1)$-step pro-nilflow. By corollary 4.6, $Y_j(Y'_j)=Y'_j$, in contradiction to $Y_j(X)$ being the j-u.c.f of X. □

6.45. **Lemma.** *Let X be a group extension of $Y_j(X)$; i.e $X=Y_j(X)\times_\sigma G$. Then $Y_{j+1}(X)$ is an abelian extension of $Y_j(X)$ by a cocycle of type j, and therefore can be given the structure of a j-step pro-nilflow.*

Proof. The proof is a straightforward generalization of lemmas 9.1, 9.2 in [FuW96] (this is done for the case $j = 3$ in [Z02b]). We outline the steps. Any ergodic component of $\bar{\triangle}_{\vec{a}}(\mu_X)$ projects onto an ergodic component of $\bar{\triangle}_{\vec{a}}(\mu_{Y_j(X)})$. The fact that $\tau_{\vec{a}}(T)$, and $T_{j+1}(T)$ commute implies that the Mackey groups associated with different ergodic component of $\bar{\triangle}_{\vec{a}}(\mu_{Y_j(X)})$ are conjugate for a.e. ergodic component (lemma 2.18). Denote $[M_{\vec{a}}]$ the conjugacy class, where the group $M_{\vec{a}} \subset G^{j+1}$. One then uses the fact that the projection of $M_{\vec{a}}$ on any j coordinates is full (i.e. G^j) to show that $[G,G]^{j+1} \subset M_{\vec{a}}$. More specifically one shows that there exists an abelian group $K_{\vec{a}}$ and homomorphisms $\psi_{\vec{a},i} : G \to K_{\vec{a}}$ so that

$$M_{\vec{a}} = \{(g_1, \dots, g_{j+1} | \psi_{\vec{a},i}(g_1) \dots \psi_{\vec{a},i}(g_{j+1}) = 1\}.$$

We return to the average in (27). By 3.3 we can replace

$$f_1 \otimes \dots \otimes f_{j+1}(y_1, g_1, \dots, y_{j+1} g_{j+1})$$

by

$$\int f_1 \otimes \dots \otimes f_{j+1}(y_1, g_1 m_1, \dots, y_{j+1}, g_{j+1} m_{j+1}) dm_{M_{\vec{a}}}(m_1, \dots, m_{j+1})$$

where $dm_{M_{\vec{a}}}$ is the Haar measure on the Mackey group $M_{\vec{a}}$. As $[G,G]^{j+1} \subset M_{\vec{a}}$ we can replace f_k, for $k = 1, \dots, j+1$, by $\int f_1(y, gg') dm_{[G,G]}(g')$. Thus $Y_j(X) \times_\rho G/[G,G]$ is characteristic for the scheme $\vec{a}$, for any $\vec{a}$.

Let $K_0 = \cap_{k,\vec{a}} \ker\psi_{\vec{a},k} = \{1\}$. Let $\tilde{G} = G/K_0$, and let $H = \tilde{G}/[\tilde{G},\tilde{G}]$. Then similarly $Y_j(X) \times_\rho H$ is characteristic for the scheme $\vec{a}$, for any $\vec{a}$. We will show that ρ is of type j. Then by 6.42 it can be given the structure of a j-step pro-nilflow, and by corollary 4.6 it the $j+1$ universal characteristic factor.

Denote $Y = Y_j(X)$. Then by equation (26)

$$\bar{\triangle}_{\vec{a}}(\mu_X) = \bar{\triangle}_{\vec{a}}(\mu_Y) \times m_H^{j+1}.$$

For each ergodic component of $\bar{\triangle}_{\vec{a}}(\mu_Y)$ the ergodic components of $\bar{\triangle}_{\vec{a}}(\mu_X)$ are given by the Mackey group $M_{\vec{a}} \subset H^{j+1}$. Above a.e. ergodic component $W_{\vec{a},y}$ (by the discussion in 4.1 the ergodic components of $\bar{\triangle}_{\vec{a}}(\mu_Y)$ are parametrized by Y) we have a H^{j+1}- extension by the cocycle

$$\tilde{\rho}_{\vec{a}} = (\rho^{(a_1)}(y_1), \rho^{(a_2)}(y_2), \dots, \rho^{(a_{j+1})}(y_{j+1})) : W_{\vec{a},y} \to H^{j+1}.$$

By Theorem 2.17 there exists a function $\varphi : W_{\vec{a},y} \to H^{j+1}$ such that

$$\varphi_{\vec{a}}(\tau_{\vec{a}}(y_1, \dots, y_{j+1})) \tilde{\rho}_{\vec{a}}(y_1, \dots, y_{j+1}) \varphi_{\vec{a}}^{-1}(y_1, \dots, y_{j+1}) \in M_{\vec{a}}$$

Applying the foregoing characterization of $M_{\vec{a}}$, there exists an abelian group $K_{\vec{a}}$ and homomorphisms $\psi_{\vec{a},i} : H \to K_{\vec{a}}$

$$\prod_{k=1}^{j+1} \psi_{\vec{a},k} \circ \rho^{(a_k)}(y_k) = F_{\vec{a}}(\tau_{\vec{a}}(y_1, \dots, y_{j+1})) F_{\vec{a}}^{-1}(y_1, \dots, y_{j+1}) \tag{45}$$

Where

$$F_{\vec{a}}(y_1,\ldots,y_{j+1})=\prod_{k=1}^{j+1}\psi_{\vec{a},i}\circ\varphi_{\vec{a},i}(y_1,\ldots,y_{j+1})\in K_{\vec{a}}.$$

Let $\chi\in\hat{K}_{\vec{a}}$. Applying χ to equation (45) we get

$$\prod_{k=1}^{j+1}\chi\circ\psi_{\vec{a},k}\circ(\rho)^{(a_k)}(y_k)=\frac{\tau_{\vec{a}}F_{\vec{a},y,\chi}(y_1,\ldots,y_{j+1})}{F_{\vec{a},y,\chi}(y_1,\ldots,y_{j+1})}. \tag{46}$$

Where $F_{\vec{a},y,\chi}:W_{\vec{a},y}\to S^1$. By ergodicity of $\tau_{\vec{a}}$ on $W_{\vec{a},y}$, $F_{\vec{a},y,\chi}$ is unique up to a constant multiple. By proposition 3.13 there is a measurable choice of $F_{\vec{a},y,\chi}$, so that equation (46) holds $\bar{\triangle}_{\vec{a}}(\mu_Y)$ a.e. Finally, as $\cap_{k=1}^{j+1}$ ker $\psi_{\vec{a},k}=\{1\}$, the characters $\chi\circ\psi_{\vec{a},k}$ where $k=1,\ldots,j+1$, $\vec{a}\in\mathbb{Z}^{j+1}$, and $\chi\in\hat{K}_{\vec{a}}$ span $\hat{H}$. □

6.46. *proof of theorem 6.1(6).*
If X is a j-step pro-nilflow then X is an abelian extension of $Y_j(X)$. By corollary 4.6, $X=Y_{j+1}(X)$. By lemma 6.45 it is an extension of $Y_j(X)$ by a cocycle of type j.

6.47. **Corollary.** *Any j-step pro-nilflow Y can be presented as a tower of abelian extensions $H_1\times_{\sigma_1}H_2\times\ldots\times_{\sigma_j}H_{j+1}$ where σ_k for $k=1,\ldots,j$ is of type k. If in this presentation H_1 is a cyclic group, and for $k>1$, H_k is a finite dimensional torus, then Y is a nilflow.*

Proof. The first part is clear. The second part follows from the construction in proposition 6.37. □

6.48. *proof of 6.1(5).*
Let

$$X=(N/\Gamma,T)=(\lim_{\leftarrow}N_i/\Gamma_i,a_i)$$

be a j-step pro-nilflow, and let W be a factor. Let $Y_j=\lim_{\leftarrow}M_i/\Lambda_i$ be the j-u.c.f of X. Then $X=Y_j\times_{\sigma_j}H$ where H is a compact abelian group, and σ_j is of type j. Let K be a compact abelian group of m.p.ts acting on X^0 and commuting with the action of T. We show that K commutes with the action of N. By corollary 2.4, any $k\in K$ induces a map from Y_j to itself, also denoted k by abuse of notation. The action of K is given by $k(y,h)=(ky,\rho_k(y,h))$. We first show that k preserves the skew product structure. As k,T commute:

$$\rho_k(T(y,h))=\rho_k(y,h)+\sigma_j(ky).$$

Denote

$$F_k(y,h)=\rho_k(y,h)-h.$$

Then

$$TF_k(y,h)-F_k(y,h)=\sigma_j(ky)-\sigma_j(y)$$

Let χ be a character of H, then

$$\frac{\chi\circ\sigma_j(ky)}{\chi\circ\sigma_j(y)}=\frac{T\chi\circ F_k(y,h)}{\chi\circ F_k(y,h)}.$$

As σ_j is of type j, $\chi\circ\sigma_j$ is lifted from M_i/Λ_i for some i. Let $p: Y_j\to M_i/\Lambda_i$ be the projection. By induction k_j commutes with the action of $M=\lim_{\leftarrow} M_i$. By proposition 6.29, as k commutes with the action of $M=\lim_{\leftarrow} M_i$, there exist functions measurable $f_{\chi,k}: M_i/\Lambda_i\to S^1$ such that

$$\frac{\chi\circ\sigma_j(ky)}{\chi\circ\sigma_j(y)}=\lambda_k\frac{Tf_{\chi,k}(py)}{f_{\chi,k}(py)}.$$

Therefore

$$\frac{\bar{f}_{\chi,k}(Tpy)\chi\circ F_k(T(y,h))}{\bar{f}_{\chi,k}(py)\chi\circ F_k(y,h)}=\lambda_k$$

Therefore $f_{\chi,k}(py)\chi\circ F_k(y,h)$ is an eigenfunction $\psi(\pi_2 y)$ (it is defined on the Kronecker factor). As $j\geq 3$, $F_k(y,h)$ depends only on y. Therefore

$$\chi\circ\rho_k(y,h)=\chi(h)\psi(\pi_2 y)f_{\chi,k}(py).$$

This implies that the action of k induces an action on N_i/Γ_i for all i, that commutes with the action of T, and by [P73] Theorem 4.3 it commutes with the action of N_i (the proof in [P73] is for $(N/\Gamma,a)$ where N is connected, but same proof holds in the case where N is generated by a and the connected component of the identity). We obtain the result inductively, using the fact that X has generalized discrete spectrum mod $\mathcal{D}$ of finite type (see [P73]), and is therefore obtained from W by a finite series of abelian extensions.

6.49. *proof of theorem 6.1(7).*

By 6.3, $Y_{j+1}(X)$ is an isometric extension of the factor $Y_j(X)$. By the discussion in 2.7, $Y_{j+1}(X)$ is of the form $Y_j(X)\times_\sigma G/L$, where G/L is a homogeneous space of a compact metric group G. By lemma 2.13 we may assume that the extension $X'=Y_j(X)\times_\sigma G$ is an ergodic group extension. As $Y_j(X)$ is a factor of $Y_j(X')$, by lemma 2.14, X' is group extension of $Y_j(X')$; i.e. $X'=Y_j(X')\times_{\sigma'} G'$. By corollary 2.4, the factor map $X'\to X$ induces a map between their $j+1$-u.c.fs. By 6.48, it is enough to show that $Y_{j+1}(X')$ has the structure f a j-step pro-nilflow. By lemma 6.45 we are done.

6.50. *proof of theorem 6.1(3c).*

By 6.41, $X=Y\times_{f_k} S^1$ can be given the structure of a j-step pro-nilflow, thus $Y_{j+1}(X)=X$. If $Y_j(X)=Y$, then by 6.46, f_k is of type j. Otherwise, Y is a proper factor of $Y_j(X)$ and therefore $Y_{j+1}(X)$ is an extension of $Y_j(X)$ by a proper closed subgroup G of S^1. By 6.46 and 6.44, G must be trivial, thus $Y_{j+1}(X)=Y_j(X)$, which is a $(j-1)$-step pro-nilflow. This implies that we can chose $f_{u,k},\lambda_{u,k}$ in equation (36), with $\lambda_{u,k}\equiv 1$ (otherwise by proposition 6.29 and the construction in 6.37, we increase the level of nilpotency). By corollary 3.9, f_k is cohomologous to a function f'_k on $Y_{j-1}(Y)$. The system

$Y_{j-1} \times_{f'_k} S^1$ is a factor of X and therefore a $j-1$-step pro-nilflow. By the induction hypothesis 1.7(3c) f'_k is of type $j-1$, therefore f_k is of type j.

6.51. *proof of theorem 6.1(3d).*

If $f : Y \to H$ is of type j then for any $\chi \in \hat{H}$ there exists $(\chi_1, \dots, \chi_l) \in \hat{H}^l$ with $\chi = \chi_k$ for some k, and $(\chi_1 \circ f, \dots, \chi_l \circ f)$ is of type $\vec{a}$ w.r.t $\bar{\triangle}_{\vec{a}}(Y)$. By 6.50, $\chi_l \circ f$ is of type j.

6.52. *proof of theorem 6.1(3e).*

By lemma 6.40, $Y \times_{fg} S^1$ can be given the structure of a pro-nilflow (it is clear from the proof of 6.15 that we can have the functions f, g lifted form the same $j-1$-step nilflow). As in 6.50, fg is of type j.

6.53. *proof of theorem 6.1(8).*

This follows from 6.49 and 4.1.

□

References

[Be87] Bergelson, V. *Weakly mixing PET*. Ergodic Theory Dynam. Systems 7 (1987), no. 3, 337-349.

[BK96] Becker, H.; Kechris, S. *The Descriptive Set Theory of Polish Group Actions*. London Mathematical Society Lecture Note Series, 232. Cambridge University Press, Cambridge, 1996.

[Bo89] Bourgain, J. *Pointwise ergodic theorems for arithmetic sets*. Inst. Hautes Études Sci. Publ. Math. No. 69 (1989), 5-45.

[CL84] Conze, J.P.; Lesigne, E. *Théorèmes ergodiques pour des mesures diagonales*. Bull. Soc. Math. France 112 (1984), no. 2, 143-175.

[CL87] Conze, J.-P.; Lesigne, E. *Sur un théorème ergodique pour des mesures diagonales*. Probabilités, 1-31, Publ. Inst. Rech. Math. Rennes, 1987−1, Univ. Rennes I, Rennes, 1988.

[CL88] Conze, J.P.; Lesigne, E. *Sur un théorème ergodique pour des mesures diagonales*. C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), no. 12, 491-493.

[Fu77] Furstenberg, H. *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*. J. Analyse Math. 31 (1977), 204-256.

[FuW96] Furstenberg, H., Weiss B. *A mean ergodic theorem for* $1/N \sum_{n=1}^{N} f(T^n x) g(T^{n^2} x)$. Convergence in ergodic theory and probability (Columbus, OH, 1993), 193-227, Ohio State Univ. Math. Res. Inst. Publ., 5 de Gruyter, Berlin, 1996.

[GOV97] Gorbatsevich, V.V., Onishchik, A.L., Vinberg, E.B. *Foundations of Lie Theory and Lie Transformation Groups*. Springer-Verlag, Berlin, 1997.

[G01] Gowers, T. A new proof of Szemerédi's theorem. GAFA 11 (2001), 465-588.

[GT04] Green B.; Tao T. The primes contain arbitrarily long arithmetic progressions. Preprint

[HK01] Host,B.; Kra, B. *Convergence of Conze-Lesigne averages*. Ergodic Theory Dynam. Systems 21 (2001), no. 2, 493-509.

[HK02a] Host,B.; Kra, B. *An odd Furstenberg-Szemerédi theorem and quasi-affine systems*. J. Analyse Math. 86 (2002), 183-220.

[HK02b] Host, B. , Kra, B. personal communication.

[HK02c] Host, B. , Kra, B. *Non-conventional ergodic averages and nilmanifolds*. To appear in the Ann. of Math. Availiable at http://www.math.psu.edu/kra/publications.html

[HK03] Host, B. , Kra, B. *Convergence of Polynomial ergodic averages*. To appear in the Isr. J. Math. Availiable at http://www.math.psu.edu/kra/publications.html

[La54] Lazard, M. *Sur les group nilpotent et les anneaux de Lie*. Ann. Sci. Ecole Norm. Sup. (3) 71 (1954), 101-190.

[L98] Leibman, A. *Polynomial sequences in groups*. J. Algebra 201 (1998), no. 1, 189-206.

[L02] Leibman, A. *Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold*. To appear in Ergodic Theory Dynam. Systems. Availiable at http://www.math.ohio-state.edu/∼leibman/preprints/

[Le84] Lesigne, E. *Résolution d'une équation fonctionnelle*. Bull. Soc. Math. France 112 (1984), no. 2, 177-196.

[Le87] Lesigne, E. *Théorèmes ergodiques ponctuels pour des mesures diagonales. Cas des systm̀es distaux*. Ann. Inst. H. Poincaré Probab. Statist. 23 (1987), no. 4, 593-612.

[Le89] Lesigne, E. *Théorèmes ergodiques pour une translation sur une nilvariete*. Ergodic Theory Dynam. Systems 9 (1989), no. 1, 115-126.

[Le93] Lesigne, E. *Equations fonctionelles, couplages de produits guaches et théorèmes ergodiques ponctuels pour mesures diagonales*. Bull. Soc. Math. France 121 (1993), no. 3, 315-351.

[Lu30] Lusin, N. *Leçons sur les Ensembles Analytiques et leurs applications*. Réimpression de l'edition de 1930. Chelsea Publishing Co., New York, 1972.

[Me90] Meiri D. *Generalized correlation series and nilpotent systems.* MSc thesis 1990. Availiable on-line at http://home.mindspring.com/ radiant4u/Math/ma.pdf

[Pe83] Peterson, C. *Ergodic Theory* Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge, 1983.

[P69] Parry, W. *Ergodic properties of affine transformations and flows on nilmanifolds*. Amer. J. Math. 91 (1969), 757-771.

[P70] Parry, W. *Dynamical systems on nilmanifolds*. Bull. London Math. Soc. 2 (1970), 37-40.

[P73] Parry, W. *Dynamical representations in nilmanifolds*. Compositio Math. 26 (1973), 159-174.

[R93] Rudolph, D. *Eigenfunctions of $T \times S$ and the Conze-Lesigne algebra*. Ergodic theory and its connections with harmonic analysis (Alexandria, 1993), 369-432, London Math. Soc. Lecture Note Ser., 205, Cambridge Univ. Press, Cambridge, 1995.

[Sh96] Shah, N. *Invariant measures and orbit closures on homogeneous spaces for actions of subgroups*. Lie groups and ergodic theory (Mumbai, 1996), 229-271, Tata Inst. Fund. Res. Stud. Math., 14, Tata Inst. Fund. Res., Bombay, 1998.

[Z02a] Ziegler, T. *A non-conventional ergodic theorem for a nilsystem*. To appear in Ergodic Theory Dynam. Systems. Availiable on-line at http://www.math.ohio-state.edu/∼tamar/

[Z02b] Ziegler, T. *Non-conventional ergodic averages*. PhD thesis. Availiable on-line at http://www.math.ohio-state.edu/∼tamar/

[Zi76] Zimmer, R. *Extensions of ergodic group actions*. Illinois J. Math. 20 (1976), no. 3, 373-409.

E-mail address: `tamar@math.ohio-state.edu`

Department of Mathematics, The Ohio State University, Columbus, Ohio